Roughness by Residuals

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$\square$ and $\diamond$

Morphology	:	$\epsilon.(\cdot)$	and	$\delta.(\cdot)$	w.r.t	binary relations
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Residuals	:	wp	and	sp	w.r.t	p.o., preorder, ...
Approximations	:	$[\![\cdot]\!]\cdot$	and	$\langle\!|\cdot|\!\rangle\cdot$	w.r.t	equivalences

Information systems and kernel relations

Features $\mathbf{F}$

$\mathcal{U}$	col	shp	edg	siz
□	w	square	4	S
⬛	b	square	4	B
▪	b	square	4	S
●	g	circle	1	S
△	w	triangle	3	B
◆	b	diamond	4	S
○	w	circle	1	S

Attributes $\mathbf{A}$

col			shp				edg			siz	
w	g	b	c	t	d	s	1	3	4	S	B
1	0	0	0	0	0	**1**	0	0	**1**	**1**	0
0	0	**1**	0	0	0	**1**	0	0	**1**	0	**1**
0	0	**1**	0	0	0	**1**	0	0	**1**	**1**	0
0	**1**	0	**1**	0	0	0	**1**	0	0	**1**	0
1	0	0	0	**1**	0	0	0	**1**	0	0	**1**
0	0	**1**	0	0	**1**	0	0	0	**1**	**1**	0
1	0	0	**1**	0	0	0	**1**	0	0	**1**	0

Information systems and kernel relations

Features $\mathbf{F}$

$\mathcal{U}$	col	shp	edg	siz
	$\mathbf{F}$			
□	`w`	`square`	4	`S`
■	`b`	`square`	4	`B`
▪	`b`	`square`	4	`S`
●	`g`	`circle`	1	`S`
△	`w`	`triangle`	3	`B`
◆	`b`	`diamond`	4	`S`
○	`w`	`circle`	1	`S`

Kernel relations

$\widetilde{col}$

□	■	▪	●	△	◆	○
1	0	0	0	**1**	0	**1**
0	**1**	**1**	0	0	**1**	0
0	**1**	**1**	0	0	**1**	0
0	0	0	**1**	0	0	0
1	0	0	0	**1**	0	**1**
0	**1**	**1**	0	0	**1**	0
1	0	0	0	**1**	0	**1**

Def. Kernel relation: $x \widetilde{f} y :\Longleftrightarrow f(x) = f(y)$

Information systems and kernel relations

Features $\mathbf{F}$

$\mathcal{U}$		$\mathbf{F}$		
	col	shp	edg	siz
□	w	square	4	S
■	b	square	4	B
▪	b	square	4	S
●	g	circle	1	S
△	w	triangle	3	B
◆	b	diamond	4	S
○	w	circle	1	S

Indiscernability

$$\tilde{\tilde{\mathbf{R}}} = \bigcap\{\tilde{col}, \tilde{shp}\}$$

□	■	▪	●	△	◆	○
1	0	0	0	0	0	0
0	**1**	**1**	0	0	0	0
0	**1**	**1**	0	0	0	0
0	0	0	**1**	0	0	0
0	0	0	0	**1**	0	0
0	0	0	0	0	**1**	0
0	0	0	0	0	0	**1**

Def. Kernel relation: $x\tilde{f}y :\Longleftrightarrow f(x) = f(y)$

Def. Indiscernability: $x\tilde{\tilde{\mathbf{R}}}y :\Longleftrightarrow x(\bigcap \mathbf{R})y$

Information systems and kernel relations

Features $\mathbf{F}$

$\mathcal{U}$	col	shp	edg	siz
	$\mathbf{F}$			
□	w	square	4	S
■	b	square	4	B
▪	b	square	4	S
●	g	circle	1	S
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◆	b	diamond	4	S
○	w	circle	1	S

$$\tilde{\tilde{\mathbf{R}}} = \bigcap\{\tilde{col}, \tilde{shp}\}$$

□	■	▪	●	△	◆	○
1	0	0	0	0	0	0
0	**1**	**1**	0	0	0	0
0	**1**	**1**	0	0	0	0
0	0	0	**1**	0	0	0
0	0	0	0	**1**	0	0
0	0	0	0	0	**1**	0
0	0	0	0	0	0	**1**

Goal: Find a smallest $\mathbf{R} \subseteq \mathrm{EquR}(\mathcal{U})$ that creates a finest $\tilde{\tilde{\mathbf{R}}}$!
($\tilde{\tilde{\mathbf{R}}} = \mathit{1}$ or $\tilde{\tilde{\mathbf{R}}} \approx T$)

Rough sets, "pointwise"

Def. Upper and Lower Approximations N.B. $R := \tilde{\tilde{\mathbf{R}}}$

$$
\begin{aligned}
[\![R]\!]s &:= \{x \in \mathcal{U} : [x]_R \subseteq s\} && (1)\\
\langle\!|R|\!\rangle s &:= \{x \in \mathcal{U} : [x]_R \cap s \neq \emptyset\}\,. && (2)
\end{aligned}
$$

Iso-/Antitony of $[\![\ ]\!]$ and $\langle\!|\ |\!\rangle$ w.r.t. set and relation arguments

$$
s \subseteq t \Longrightarrow \begin{array}{ccc} \langle\!|R|\!\rangle s & \subseteq & \langle\!|R|\!\rangle t \\ [\![R]\!]s & \subseteq & [\![R]\!]t \end{array} \text{ but } P \subseteq R \Longrightarrow \begin{array}{ccc} \langle\!|P|\!\rangle s & \subseteq & \langle\!|R|\!\rangle s \\ [\![P]\!]s & \supseteq & [\![R]\!]s \end{array}
$$

Duality (as desired)

$$
[\![R]\!]\overline{s} = \overline{\langle\!|R|\!\rangle s}. \qquad (3)
$$

Rough sets by characteristic functions

We represent $s \subseteq \mathcal{U}$ by its characteristic relation $\dot{s} : \mathcal{U} \to \mathbf{2}$.
We observe (pointwise):

$$x \in [\![R]\!]s \iff [x]_R \subseteq s$$

Def. Upper and Lower Approximations, again.

$$[\![R]\!]s \quad := \quad \langle\, R \backslash\!\backslash \dot{s} \,|\, \{\mathbf{1}\} \tag{4}$$

$$\langle\!|R|\!\rangle s \quad := \quad \langle\, R \backslash\!\backslash \overline{\dot{s}} \,|\, \{\mathbf{0}\} \tag{5}$$

where

$$\begin{array}{lllllllll} \mathrm{RR}: & P \backslash\!\backslash Q & := & \overline{P^{\smile} ; \overline{Q}} & \text{or} & R \subseteq P \backslash\!\backslash Q & \iff & P ; R \subseteq Q \\ \mathrm{LR}: & Q /\!\!/ P & := & \overline{\overline{Q} ; P^{\smile}} & \text{or} & R \subseteq Q /\!\!/ P & \iff & R ; P \subseteq Q \end{array}$$

with R being the biggest solution of the respective inequalities.

Rough sets by characteristic functions

We represent $s \subseteq \mathcal{U}$ by its characteristic relation $\dot{s} : \mathcal{U} \to \mathbf{2}$.
We observe (pointwise):

$$x \in [\![R]\!] s \iff xRy \longrightarrow y \in s$$

Def. Upper and Lower Approximations, again.

$$[\![R]\!] s \quad := \quad \langle\, R \backslash\!\!\backslash \dot{s} \,|\, \{\mathbf{1}\} \qquad (4)$$

$$\langle\!| R |\!\rangle s \quad := \quad \langle\, R \backslash\!\!\backslash \overline{\dot{s}} \,|\, \{\mathbf{0}\} \qquad (5)$$

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$$x \in [\![R]\!]s \iff x\overline{R}y \vee y\dot{s}\mathbf{1}$$

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We represent $s \subseteq \mathcal{U}$ by its characteristic relation $\dot{s} : \mathcal{U} \to \mathbf{2}$.
We observe (pointwise):

$$x \in [\![R]\!]s \iff \neg(y\overline{\dot{s}}\mathbf{1} \wedge xRy)$$

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Def. Upper and Lower Approximations, again.

$$\begin{aligned} [\![R]\!] s &:= \langle\, R \backslash\!\!\backslash \dot{s} \mid \{\mathbf{1}\} \quad (4) \\ \langle\!|R|\!\rangle s &:= \langle\, R \backslash\!\!\backslash \overline{\dot{s}} \mid \{\mathbf{0}\} \quad (5) \end{aligned}$$

where

$$\begin{array}{lllllll} \mathrm{RR}: & P \backslash\!\!\backslash Q & := & \overline{P^{\smile} ; \overline{Q}} & \text{or} & R \subseteq P \backslash\!\!\backslash Q & \iff & P ; R \subseteq Q \\ \mathrm{LR}: & Q /\!\!/ P & := & \overline{\overline{Q} ; P^{\smile}} & \text{or} & R \subseteq Q /\!\!/ P & \iff & R ; P \subseteq Q \end{array}$$

with R being the biggest solution of the respective inequalities.

Example

$\dot{c}$	**0**	**1**
□	1	0
■	1	0
▪	0	**1**
●	0	**1**
△	0	**1**
◆	1	0
○	1	0

(columns: $\mathbf{2}$)

$\widetilde{shp}$	□	■	▪	●	△	◆	○
□	**1**	**1**	1	0	0	0	0
■	**1**	**1**	1	0	0	0	0
▪	1	1	$\cancel{1}$	0	0	0	0
●	0	0	0	$\cancel{1}$	0	0	1
△	0	0	0	0	**1**	0	0
◆	0	0	0	0	0	1	0
○	0	0	0	**1**	0	0	**1**

(columns: $\mathcal{U}$)

$\widetilde{shp} \backslash\!\backslash \dot{c}$	**0**	**1**
□	0	0
■	0	0
▪	0	0
●	0	0
△	0	1
◆	1	0
○	0	0

(columns: $\mathbf{2}$)

Therefore, $\langle\, \widetilde{shp} \backslash\!\backslash \dot{c} \mid \{\mathbf{1}\} = \langle\, \overline{\widetilde{shp}^{\smile} \mathbin{;} \bar{\dot{c}}} \mid \{\mathbf{1}\} = \{\triangle\} = [\![\widetilde{shp}]\!]\,\{\blacksquare, \bullet, \triangle\}$.

Rough sets by subidentities

We represent $s \subseteq \mathcal{U}$ by its subidentity $\mathsf{S} = \mathit{1} \cap (s \times s)$ and define $S^- := \mathit{1} \cap \overline{S}$. We observe:

$$x \in [\![R]\!]s \iff [x]_R \subseteq \mathsf{S}.\mathcal{U}$$

Def. Upper and Lower Approximations, again.

$$[\![R]\!]s \quad := \quad \mathrm{dom}\,(\, R \backslash\!\backslash \mathsf{S}\,) \tag{6}$$

$$\langle\!|R|\!\rangle s \quad := \quad \overline{\mathrm{dom}\,(\, R \backslash\!\backslash \mathsf{S}^-\,)} \tag{7}$$

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$$x \in [\![R]\!] s \Longleftrightarrow \forall y : xRy \longrightarrow y\mathsf{S}y$$

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$$\langle\!| R |\!\rangle s := \overline{\mathrm{dom}\,(\, R \backslash\!\backslash \mathsf{S}^-\,)} \tag{7}$$

Def. Approximations of classifications

$$[\![R]\!] \mathcal{U}/Q := \{[\![R]\!] c : c \in \mathfrak{c}\} = \{\mathrm{dom}\,(\, R \backslash\!\backslash \mathsf{C}\,) : c \in \mathfrak{c}\} \tag{8}$$

where $\mathfrak{c} = \mathcal{U}/Q$ is a *target classification*.

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$$x \in [\![R]\!] s \Longleftrightarrow \forall y : \neg(y\overline{\mathsf{S}}y \wedge xRy)$$

Def. Upper and Lower Approximations, again.

$$\begin{aligned} [\![R]\!] s &:= \operatorname{dom}(\, R \backslash\!\!\backslash \mathsf{S}\,) && (6) \\ \langle\!|R|\!\rangle s &:= \overline{\operatorname{dom}(\, R \backslash\!\!\backslash \mathsf{S}^{-}\,)} && (7) \end{aligned}$$

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We represent $s \subseteq \mathcal{U}$ by its subidentity $\mathsf{S} = \mathit{1} \cap (s \times s)$ and define $S^- := \mathit{1} \cap \overline{S}$. We observe:

$$x \in [\![R]\!] s \iff \forall y : x\, \overline{R^\smile ; \overline{\mathsf{S}}}\, y$$

Def. Upper and Lower Approximations, again.

$$[\![R]\!] s \quad := \quad \mathrm{dom}\,(\, R \backslash\!\backslash \mathsf{S}\,) \tag{6}$$

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$$x \in [\![R]\!] s \Longleftrightarrow \forall y : \, x \, R \backslash\!\!\backslash \mathsf{S} \, y$$

Def. Upper and Lower Approximations, again.

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$$x \in [\![R]\!] s \iff x \in R \backslash\!\backslash \mathsf{S} \,.\, \mathcal{U}$$

Def. Upper and Lower Approximations, again.

$$[\![R]\!] s := \operatorname{dom}(R \backslash\!\backslash \mathsf{S}) \quad (6)$$

$$\langle\!| R |\!\rangle s := \overline{\operatorname{dom}(R \backslash\!\backslash \mathsf{S}^{-})} \quad (7)$$

Def. Approximations of classifications

$$[\![R]\!]\mathcal{U}/Q := \{[\![R]\!]c : c \in \mathfrak{c}\} = \{\operatorname{dom}(R \backslash\!\backslash \mathsf{C}) : c \in \mathfrak{c}\} \quad (8)$$

where $\mathfrak{c} = \mathcal{U}/Q$ is a *target classification*.

Rough sets by KAD

Preimages, domains and tests

With KAT we are given

$$\langle R| \quad := \quad \min\{X \in \mathsf{U} : R \subseteq X ⨾ R\} \tag{9}$$

$$-\langle R| \quad := \quad \max\left\{X \in \mathsf{U} : X ⨾ R \subseteq \overline{C}\right\} \tag{10}$$

Here, $C = \mathbb{T}$ (i.e. not a relative, but absolute complement).
Then, by domain laws,

$$\langle R|\,\mathsf{S} = \langle\, R⨾\mathsf{S}\,| = \langle\,\overline{R\backslash\!\backslash \mathsf{S}^{-}}\,| = \langle\!|R|\!\rangle s \tag{11}$$

and, canonically,

$$[R|\,\mathsf{T} := -\langle R|\,\mathsf{T}^{-} = \langle\, R\backslash\!\backslash \mathsf{T}\,| = [\![R]\!]t. \tag{12}$$

Rough sets by GC

... are for free.

With $[\![\cdot]\!]$ / $\langle\!|\cdot|\!\rangle$ being defined by $[\cdot|\,\cdot$ / $\langle\cdot|\,\cdot$, and reading sets as subidentities,

$$\langle R|\, s \subseteq t \iff \langle R|\, t^- \subseteq s^- \iff s \subseteq -\langle R|\, t^- \iff s \subseteq [R|\, t. \quad (13)$$

What we get for free

- $[\![R]\!]\overline{s} = \overline{\langle\!|R|\!\rangle s}$ proving (3).
- $[\![R]\!]s = s \iff \langle\!|R|\!\rangle s = s$ proving (17, 18) in the paper.
- ...
- a nice metaphor: "RST is just a GC, with only equivalences".

Positive regions and refinement

How good is R to ...

- describe a classification $\mathcal{U}/Q$ (or s/Q)...
- compared to a relation P?

Def. Positive regions

$$[\![R \lessdot P]\!]s/Q := \bigcup_{c \in s/Q} [\![R \lessdot P]\!]c \ := \ \bigcup_{c \in s/Q} \bigcup_{t \in c/P} [\![R]\!]t. \tag{14}$$

$[\![R \lessdot P]\!]s/Q$ is the *largest set of elements* $x \in s$ s.t. $[x]_R \subseteq [x]_P \subseteq [x]_Q$.

Positive regions and refinement

How good is R to ...

- describe a classification $\mathcal{U}/Q$ (or s/Q)...
- compared to a relation P?

Def. Positive regions

$$[\![R \lessdot P]\!] s/Q := \bigcup_{c \in s/Q} [\![R \lessdot P]\!] c \ := \bigcup_{c \in s/Q} \bigcup_{t \in c/P} [\![R]\!] t. \qquad (14)$$

Positive regions by residuals

$$[\![R \lessdot P]\!] s/Q = \langle \, (\, R \mathbin{\backslash\!\!\backslash} P \,) \mathbin{\backslash\!\!\backslash} Q \mid s = \mathrm{dom} \, (\, (\, R \mathbin{\backslash\!\!\backslash} P \,) \mathbin{\backslash\!\!\backslash} Q \,;\! \mathsf{S} \,) \qquad (15)$$

$$\begin{array}{lllll} \text{s.t.} & [\![R]\!] s & = \mathrm{dom} \, (\, (\, R \mathbin{\backslash\!\!\backslash} \widetilde{s} \,) \mathbin{\backslash\!\!\backslash} \mathsf{S} \,) & = \mathrm{dom} \, (\, R \mathbin{\backslash\!\!\backslash} \mathsf{S} \,) \\ \text{but} & \bigcup [\![R]\!] (s/Q) = & R \mathbin{\backslash\!\!\backslash} Q \,.\, s & = \mathrm{dom} \, (\, R \mathbin{\backslash\!\!\backslash} Q \,;\! \mathsf{S} \,) \end{array}$$

R is better than P, if...

Def. Refinement

R refines P w.r.t H on s/Q, iff

$$R \underset{s/Q}{\overset{H}{\succeq}} P \quad :\Longleftrightarrow \quad [\![R \lessdot H]\!](s/Q) \supseteq [\![P \lessdot H]\!](s/Q) \tag{16}$$

$$\Longleftrightarrow \quad \mathrm{dom}\,(\,(\,(\,R \setminus\!\!\setminus H\,) \setminus\!\!\setminus Q\,) ; \mathsf{S}\,) \supseteq \mathrm{dom}\,(\,(\,(\,P \setminus\!\!\setminus H\,) \setminus\!\!\setminus Q\,) ; \mathsf{S}\,)$$

Def. $\mathbf{R}$ is a (H-relative) reduct of $\mathbf{P}$ (on s/Q), if...

1. $\mathbf{R} \subseteq \mathbf{P}$ (hence $\tilde{\tilde{\tilde{\mathbf{P}}}} \subseteq \tilde{\tilde{\tilde{\mathbf{R}}}}$ and $\mathbf{P} \succeq \mathbf{R}$)
2. $\mathbf{R}$ is a *largest* subset of $\mathbf{P}$ s.t. $\mathbf{R} \succeq \mathbf{P}$.

$\mathrm{Red}(\mathbf{P})$ is the set of all reducts of $\mathbf{P}$; $\mathrm{Cor}(\mathbf{P}) := \bigcap \mathrm{Red}(\mathbf{R})$.

R is better than P, if...

Def. Refinement

R refines P w.r.t H on s, iff

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$$R \overset{H}{\succeq} P \quad :\Longleftrightarrow \quad [\![R \lessdot H]\!]\mathcal{U} \supseteq [\![P \lessdot H]\!]\mathcal{U} \qquad (16)$$
$$\Longleftrightarrow \quad \operatorname{dom}(\, R \bbslash H \,) \supseteq \operatorname{dom}(\, P \bbslash H \,)$$

Def. $\mathbf{R}$ is a (H-relative) reduct of $\mathbf{P}$ (on s/Q), if...

1. $\mathbf{R} \subseteq \mathbf{P}$ (hence $\tilde{\tilde{\mathbf{P}}} \subseteq \tilde{\tilde{\mathbf{R}}}$ and $\mathbf{P} \succeq \mathbf{R}$)
2. $\mathbf{R}$ is a *largest* subset of $\mathbf{P}$ s.t. $\mathbf{R} \succeq \mathbf{P}$.

$\mathrm{Red}(\mathbf{P})$ is the set of all reducts of $\mathbf{P}$; $\mathrm{Cor}(\mathbf{P}) := \bigcap \mathrm{Red}(\mathbf{R})$.

We need to make a few points

It is nice to know that $\mathbf{R} \in \mathrm{Red}(\mathbf{P})$,
but the definition does not help <u>finding</u> such $\mathbf{R}$.

<u>**Def.**</u> "$\underset{\cdot}{s}$" is an arbitrarily chosen, fixed element (point) of s.

Sets and Partitions

- Sets
  - characteristic relations: $\dot{s} : \mathcal{U} \to \mathbf{2}$ $\qquad s = \langle \dot{s} |\, 1$
  - subidentities: $\mathsf{S} := \mathit{1} \cap (s \times s)$ $\qquad s = \langle \mathsf{S} | \mathcal{U}$
- Classifications
  - quotients s/R: $\qquad s/R = \{ \langle R |\, x : x \in s \}$
  - pointwise representation: $r \subseteq s$ $\qquad s/R = \{ [\underset{\cdot}{c}] : \underset{\cdot}{c} \in r \}$

Concrete Tasks

... sadly require concrete data (points)

Goal: Construct (efficiently) some $\mathbf{R}$ in $\mathrm{Red}(\mathbf{P})$

Gedankenexperiment $(\mathbf{Q} := \mathbf{R} - \{R\})$

Suppose, $R \in \mathrm{Cor}(\mathbf{P})$.

- Then, $R \in \mathbf{R}$ for all $\mathbf{R} \in \mathrm{Red}(\mathbf{P})$.
- Hence, $\mathbf{R} \overset{\mathbf{P}}{\succ} \mathbf{Q}$ (strictly!).
- By definition, $\exists x : x \notin [\![(\mathbf{Q} \lessdot \mathbf{P}]\!]s \subset [\![\mathbf{R} \lessdot \mathbf{P}]\!]s \ni x$.
- Then, $[x]_{\overset{\approx}{\mathbf{R}}} \subseteq [x]_{\overset{\approx}{\mathbf{P}}}$ but $[x]_{\mathbf{Q}} \not\subseteq [x]_{\overset{\approx}{\mathbf{P}}}$.
- Therefore, there is a representation r of $s/\overset{\approx}{\mathbf{P}}$, s.t.

$$\exists \underset{\cdot}{r}_1, \underset{\cdot}{r}_2 : \underset{\cdot}{r}_1 \overline{R} \underset{\cdot}{r}_2 \wedge [\underset{\cdot}{r}_1]_{\overset{\approx}{\mathbf{Q}}} = [\underset{\cdot}{r}_2]_{\overset{\approx}{\mathbf{Q}}} \tag{17}$$

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- By definition, $\exists x : x \notin [\![(\mathbf{Q} \lessdot \mathbf{P}]\!] s \subset [\![\mathbf{R} \lessdot \mathbf{P}]\!] s \ni x$.
- Then, $[x]_{\tilde{\tilde{\mathbf{R}}}} \subseteq [x]_{\tilde{\tilde{\mathbf{P}}}}$ but $[x]_{\mathbf{Q}} \not\subseteq [x]_{\tilde{\tilde{\mathbf{P}}}}$.
- Therefore, there is a representation r of $s/\tilde{\tilde{\mathbf{P}}}$, s.t.

$$\exists \underset{.}{r}_1, \underset{.}{r}_2 : \underset{.}{r}_1 \overline{R} \underset{.}{r}_2 \wedge \forall Q \in \mathbf{R} : Q \neq R \longrightarrow \underset{.}{r}_1 Q \underset{.}{r}_2 \tag{17}$$

A visual example

"Discernability matrices"

$\Delta\mathbf{P}$	□	■	●	△	◆	○
□		ZC	CES	ZES	CES	ES
■			ZCES	CES	ZS	ZCES
●				ZCES	CES	C
△					ZCES	ZES
◆						CES
○						

1. No entry is nil — hence, $\tilde{\tilde{\mathbf{P}}} = \bigcap \left\{ \widetilde{siz}, \widetilde{col}, \widetilde{edg}, \widetilde{shp} \right\} = 1$
2. In any $\mathbf{R} \in \mathrm{Red}(\mathbf{P})$, $\tilde{\tilde{shp}}$ is indispensable: it is essential
3. $\widetilde{col} \in \mathrm{Cor}(\mathbf{P})$

Discernability of representatives

- If we have to make points, try making as few as possible.
- Speeding up exhaustive pointwise processes.

A (last) motivational example $\mathcal{U} = \{\square, \blacksquare, \bullet, \triangle, \blacklozenge, \bigcirc\}$

1. Let $\mathbf{P} = \left\{\tilde{col}, \tilde{shp}\right\}$.
2. Is $\tilde{col}$ indispensable in $\mathbf{P}$ w.r.t. *1*?
   2.1 Choose wisely $r_{\mathbf{P}-\left\{\tilde{col}\right\}} = r_{\tilde{shp}} = \{\blacksquare, \blacklozenge, \bigcirc, \triangle\}$.
   2.2 Consider $\blacksquare$: Then, $[\blacksquare]_{\tilde{\mathbf{P}}} = \{\blacksquare\}$.
   2.3 But: $\blacksquare \notin [\![\left\{\tilde{shp}\right\} \lessdot \mathbf{P}]\!]\mathcal{U}$
   because $[\blacksquare]_{\tilde{shp}} = \{\blacksquare, \square\} \not\subseteq \{\blacksquare\} = [\blacksquare]_{\tilde{\mathbf{P}}}$
3. Hence, $\tilde{col}$ is essential

Finding Reducts (Skowron's exhaustive approach)

1. (Compute $\mathfrak{c} := s/Q$.)
2. (Compute $h := [\![\mathbf{R} \lessdot Q]\!]s$.)

4. For every $x, y \in \mathcal{U}$, compute $\left\{R : R \in \mathbf{R} \wedge x\overline{R}y\right\}$
5. $\mathrm{Cor}_{\mathfrak{c}}(\mathbf{R}) := \left\{R : \exists x, y \in r : x\overline{\bigcap \mathbf{R} - \{R\}}y\right\}$
6. For every $\mathbf{P} \subseteq \mathbf{R} - \mathrm{Cor}_{\mathfrak{c}}(\mathbf{R})$:

   $\mathbf{Q} := \mathbf{P} \cup \mathrm{Cor}_{\mathfrak{c}}(\mathbf{R})s$ is a Q-reduct of $\mathbf{R}$ w.r.t. s, iff:
   $\mathbf{Q} \overset{Q}{\succeq}_s \mathbf{R}$ and $\mathbf{Q}$ is not a superset of any other reduct.

Finding Reducts (A not so exhaustive approach)

1. (Compute $\mathfrak{c} := s/Q$.)
2. (Compute $h := [\![\mathbf{R} \lessdot Q]\!]s$.)
3. Guess a suitable $r := \{\underset{\cdot}{c} : c \in h/Q\}$.
4. For every $x, y \in r$, compute $\left\{R : R \in \mathbf{R} \wedge x\overline{R}y\right\}$
5. $\mathrm{Cor}_{\mathfrak{c}}(\mathbf{R}) := \left\{R : \exists x, y \in r : x\overline{\bigcap \mathbf{R} - \{R\}}y\right\}$
6. For every $\mathbf{P} \subseteq \mathbf{R} - \mathrm{Cor}_{\mathfrak{c}}(\mathbf{R})$ ordered by voodoo:

   $\mathbf{Q} := \mathbf{P} \cup \mathrm{Cor}_{\mathfrak{c}}(\mathbf{R})s$ is a Q-reduct of $\mathbf{R}$ w.r.t. s, iff:
   $\mathbf{Q} \overset{Q}{\succeq}_s \mathbf{R}$ and $\mathbf{Q}$ is not a superset of any other reduct.

Conclusion

Summary

- Rough sets by residuals
- Rough sets by GC
- An algorithm for reduct construction

Prospects

- Rough sets by KAT (if not done yet)
- Rough sets by morphology (if not done yet)
- Rough sets by formal concept analysis (if not done yet)
- … automatically create a searchable space of logic programs with $\top$ being a program s.t. $P \not\vdash E^-$ and $\perp$ s.t. $Q \vdash E^+$ with $\models\!\approx$ as p.o. (not done yet)

$$you \in \mathrm{dom}\,(\,Thanks \mathbin{⨾} \mathsf{ATTENTION}\,)\,.$$