Towards Antichain Algebra

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RAMiCS 2015

Overview

- maximal objects in a subset of partial order are ones that have no objects strictly above them
- hence they are pairwise incomparable, i.e., form an antichain
- maximal objects play an important role in many algorithms
- since we are interested in algebraic program derivation
- we present an algebra of (strict-)orders and antichains
- an approximation relation between antichains induces a semilattice
- the maxima operator can be viewed as a closure operator in an associated pre-ordered set
- this finally yields a characterisation of antichains in terms of a Galois connection

sample application:

- preference databases
- user specifies her preferences as a strict-order
- BMO (best matches only) semantics returns the maximal objects, because these meet user wishes best
- we algebraically derive the standard Block-Nested Loop (BNL) algorithm for computing the maxima
- approximation order reflects the steps taken by the BNL algorithm
- antichain algebra can be used to improve the efficiency

Strict-Orders and Maxima Algebraically

concrete	abstract
relation between objects	semiring element a
composition ;	semiring multiplication $\cdot$
identity relation	multiplicative semiring unit 1
union	semiring addition $+$
inclusion	subsumption order $a \le b \Leftrightarrow a + b = b$
sets of objects	tests $p \le 1$
single objects	atomic tests
inverse image	$\|a\rangle\, p$

- element a is d(iamond)-transitive if $\forall\, p : |a \cdot a\rangle\, p \leq |a\rangle\, p$
- more liberal than stipulating $a \cdot a \leq a$
- for relations a both formulations coincide
- a is d-irreflexive if for all atomic $x : x \cdot |a\rangle\, x \leq 0$
- strict-order: d-transitive and d-irreflexive element

- best or maximal objects w.r.t. element a and test p:

$$a \rhd p \;=_{df}\; p - |a\rangle\, p$$

- interpretation for (preference) strict-order a:
- $|a\rangle\, p$, the inverse image of p under a, is the set of objects a-dominated by some object in p
- thus $p - |a\rangle\, p$ are the non-dominated, hence maximal objects in p

some useful properties:

1. $a \triangleright 0 = 0$
2. $a \triangleright 1 = \neg \ulcorner a$
3. $\ulcorner b \leq \ulcorner a \;\Leftrightarrow\; a \triangleright 1 \leq b \triangleright 1$
4. $a \triangleright p \leq p$
5. $\boxed{a \triangleright (a \triangleright p) = a \triangleright p}$
6. $(a + b) \triangleright p = (a \triangleright p) \cdot (b \triangleright p)$.
7. $b \leq a \;\Rightarrow\; a \triangleright p \leq b \triangleright p$, i.e., $\triangleright$ is antitone in its first argument
8. $1 \leq a \;\Rightarrow\; a \triangleright p = 0$

- so far no special properties of strict-orders required
- for further laws need an assumption that “enough” maximal objects exist
- expressed by requiring every non-maximal object to be dominated by some maximal one
- always satisfied if set of all objects is finite (as in databases)
- infinite case closely related with noetherity (see below)

- element a is called normal if $\forall\, p : |a\rangle\, p \leq |a\rangle\, (a \triangleright p)$
- meaning: every object dominated by some p-object is also dominated by a maximal p-object
- equivalent to $\forall\, p : |a\rangle\, p = |a\rangle\, (a \triangleright p)$

- element a is noetherian if, for all tests p,

$$a \triangleright p \leq 0 \Rightarrow p \leq 0 \ .$$

- by contraposition and leastness of 0 equivalent to

$$p \neq 0 \Rightarrow a \triangleright p \neq 0$$

- means that every non-empty p contains at least one maximal object (dual of the usual well-foundedness condition)
- in the relational case therefore also equivalent to the absence of infinitely ascending chains

Theorem

- if a is noetherian then for any $q \in \mathsf{test}(S)$ we have $q \leq |a^*\rangle\,(a \triangleright q)$, i.e., all points in q are a^*-dominated by points in $a \triangleright q$
- every noetherian and d-transitive element is normal
- every normal element is noetherian and d-transitive

- important application:
- a normal $\Rightarrow \;\; a \triangleright (p+q) = a \triangleright (a \triangleright p + a \triangleright q)$
- paves the way for a distributed computation of maxima:
- for disjoint p and q the calculations of $a \triangleright p$ and $a \triangleright q$ are independent
- law generalises from $+$ to arbitrary existing suprema in the set of tests

Antichains

- antichain: set of mutually incomparable objects
- equivalently, a set is an antichain if it equals its maxima set
- algebraic characterisation:
- for a semiring element a, a test p is an a-antichain if $p = a \triangleright p$
- $\mathrm{AC}(a)$: set of all a-antichains
- $0 \in \mathrm{AC}(a)$ for every a
- for d-irreflexive a every atomic test is an antichain
- $\mathrm{AC}(a)$ is downward closed, i.e.,

$$p \in \mathrm{AC}(a) \land q \leq p \Rightarrow q \in \mathrm{AC}(a)$$

Lattice Structure of Antichains

- we now exhibit a lattice structure on the set of antichains
- first we define an approximation relation
- test p is improved by test q, in symbols $p \sqsubseteq q$, if q results from removing some objects of p that are dominated by q-objects
- and possibly adding others that are not dominated by p-objects
- formally,

$$p \sqsubseteq q \Leftrightarrow_{df} p - |a\rangle\, q \leq q \wedge q \cdot |a\rangle\, p \leq 0$$

- by Boolean algebra and distributivity, equivalently

$$p \sqsubseteq q \Leftrightarrow p \leq |a+1\rangle\, q \wedge q \cdot |a\rangle\, p \leq 0$$

properties:

- $\forall\, p \in \mathsf{test}(S) : 0 \sqsubseteq p$.
- $\sqsubseteq$ is reflexive precisely on $\mathrm{AC}(a)$, i.e., $p \sqsubseteq p \Leftrightarrow p \in \mathrm{AC}(a)$
- $\sqsubseteq$ is antisymmetric
- if a is d-transitive, then for antichains the second conjunct in the definition of $\sqsubseteq$ is implied by the first one, i.e., for $p, q \in \mathrm{AC}(a)$ we have $p \sqsubseteq q \Leftrightarrow p \leq |a+1\rangle\, q$
- if a is d-transitive then $\sqsubseteq$ is transitive and hence a partial order on $\mathrm{AC}(a)$.
- If a is normal then $p \sqsubseteq a \triangleright p$

Theorem

- $a \triangleright$ transforms all $\leq$-suprema in test(S) into $\sqsubseteq$-suprema in $\mathrm{AC}(a)$
- $a \triangleright$ is isotone w.r.t. $\leq$ and $\sqsubseteq$, i.e.,

$$\forall\, p, q \in \mathsf{test}(S) : p \leq q \;\Rightarrow\; a \triangleright p \sqsubseteq a \triangleright q$$

- $\mathrm{AC}(a)$ is an upper semilattice with $p \sqcup q = a \triangleright (p + q)$ and $0 \sqcup p = p$
- if $(S, \leq)$ is a quantale then $\mathrm{AC}(a)$ is a complete lattice with $\bigsqcup_{\sqsubseteq} A = a \triangleright (\Sigma A)$, where Σ is the supremum operator on $(S, \leq)$
- $a \triangleright$ preserves $\sqcup$ on $\mathrm{AC}(a)$
- $a \triangleright$ is also isotone w.r.t. $\sqsubseteq$ and $\sqsubseteq$ on arbitrary tests:

$$\forall\, p, q \in \mathsf{test}(S) : p \sqsubseteq q \;\Rightarrow\; a \triangleright p \sqsubseteq a \triangleright q$$

Maxima as a Closure Operator

a closure operator on a partially ordered set $(L, \leq)$ is a total function $f : L \to L$ with the following properties:

$$x \leq f(x) \qquad \text{(extensivity)}$$

$$x \leq y \Rightarrow f(x) \leq f(y) \qquad \text{(isotony)}$$

$$f(f(x)) = f(x) \qquad \text{(idempotence)}$$

- by earlier properties $a \triangleright$ satisfies all three properties of a closure operator w.r.t. $\sqsubseteq$
- unfortunately, however, $\sqsubseteq$ is not even a preorder on test(S), since reflexivity holds exactly on $\mathrm{AC}(a)$
- to remedy this, we define another comparison relation on test(S):
- $p \preceq_a q \;\Leftrightarrow_{df}\; a \triangleright p \sqsubseteq a \triangleright q$
- then $\preceq$ is a preorder, but not a partial order
- we have $p \preceq q \wedge q \preceq p \;\Leftrightarrow\; a \triangleright p = a \triangleright q$
- finally, $p \leq q \;\Rightarrow\; p \preceq q$
- with the definition of $\preceq$ we can now actually view $a \triangleright$ as a closure operator by carrying the notion over to preorders

A Galois Connection for the Maxima Operator

since the maxima operator is a closure operator, we can use a well-known result concerning Galois connections, again adapted to the case of preorders rather than partial orders

- consider two preorders $(A, \leq_A)$ and $(B, \leq_B)$ and total functions $F : A \to B$ and $G : B \to A$
- the pair (F, G) is called a Galois connection (GC) between A and B iff

$$\forall\, x \in A : \forall\, y \in B : F(x) \leq_B y \Leftrightarrow x \leq_A G(y)$$

- F is called the lower, G the upper adjoint of the GC the following result is well known for the case of partial orders; we adapt it to preorders

every closure operator $H : L \to L$ induces the following Galois connection between L and $H(L)$:

$$H(x) \leq y \Leftrightarrow x \leq \iota(y)$$

where ι is the embedding of $H(L)$ into L, i.e., $\iota(y) = y$ for $y \in H(L)$

hence for $p \in \mathsf{test}(S)$ and $q \in \mathrm{AC}(a)$ we have the Galois connection

$$a \triangleright p \preceq q \;\Leftrightarrow\; p \preceq \iota(q)$$

as a lower adjoint therefore the $a \triangleright$ operator preserves all existing $\preceq$-suprema

this nicely rounds off the small collection of preservation results in the main theorem

Application

- we now sketch an algebraic, calculational derivation of the standard BNL algorithm for computing maximal objects
- we assume that the test algebra of the underlying semiring is finite and hence atomic, i.e.,
- every test is the sum of the atoms below it
- let test r represents all available tuples in a database and a be a fixed strict-order representing a preference relation
- the task is to compute $a \triangleright r$, i.e., a test representing the set of all a-maximal objects in r

standard approach:

- make a constant of the specification into a parameter
- calculate an inductive or recursive version of the generalised specification
- here: make r into a parameter called u
- hence for test u we define the function $ma(u)$ that computes the maxima of u w.r.t. preference a as

$$ma(u) =_{df} a \triangleright u$$

aim

- develop a recursive version of the function ma by induction on the size of the parameter u
- by the finiteness and atomicity of the test algebra, the size $|u|$ of u can be defined as the cardinality of the set of atoms below u.

base case $|u| = 0$

- then $u = 0$
- hence $ma(0) = 0 - |a\rangle\, 0 = 0$.

inductive case: choose an atomic test $x \leq u$ and set $v =_{df} u - x$

$$
\begin{array}{ll}
 & ma(u) \\
= & \quad \{\![\ \text{unfold } ma\ ]\!\} \\
 & a \triangleright (x + v) \\
= & \quad \{\![\ \text{max-additivity}\ ]\!\} \\
 & a \triangleright (a \triangleright x + a \triangleright v) \\
= & \quad \{\![\ \text{d-irreflexivity of } a\text{, atomicity of } x\ ]\!\} \\
 & a \triangleright (x + a \triangleright v) \\
= & \quad \{\![\ \text{fold } ma\ ]\!\} \\
 & a \triangleright (x + ma(v))
\end{array}
$$

now, since $ma(v) = a \triangleright v$ is an antichain, we define an auxiliary function

$$inc(x, p) \ =_{df} \ a \triangleright (x + p) \ = \ x \sqcup p$$

where x is an atomic test and p an antichain

then we can continue the previous derivation to obtain $ma(u) = inc(x, ma(v))$

altogether, we have derived the recursion

$$ma(u) = \text{if } u = 0 \text{ then } 0$$
$$\text{else choose atom } x \le u \text{ in}$$
$$inc(x, ma(u - x))$$

- the original task is now solved using the call $ma(r)$
- by the main theorem we have $p \sqsubseteq inc(x, p)$
- hence the BNL algorithm produces a $\sqsubseteq$-ascending chain of antichains ending with the $\sqsubseteq$-largest antichain $a \triangleright r$

- now we apply the algebra to bring the function ma into tail-recursive form,
- as a preparation for transliterating it into loop form
- essential observation: the expression in the recursive case is $inc(x, ma(u-x)) = x \sqcup ma(u-x)$ and
- $\sqcup$ as a supremum operator is associative and has the $\sqsubseteq$-least element 0 as its neutral element
- we define an auxiliary function $mat(p, u) =_{df} p \sqcup ma(u)$ with an additional parameter p that will accumulate the end result
- by neutrality of 0 we can solve the original task as $ma(u) = mat(0, u)$

we calculate a recursive version of mat from the one for ma by the usual re-bracketing technique

- in the termination case $u = 0$ we obtain $mat(p, 0) = p \sqcup 0 = p$
- In the recursive case for $u \neq 0$ we get by unfolding, the main theorem, associativity of $\sqcup$ and folding

  $$mat(p, u) = p \sqcup inc(x, ma(u - x)) = p \sqcup (x \sqcup ma(u - x)) = (p \sqcup x) \sqcup ma(u - x) = mat(p \sqcup x, u - x) \ ,$$

  which is a tail-recursive call

- in the paper we similarly calculate a recursive version of the function $inc(x, p)$
- parameter p is frequently called the (working) window
- it contains candidates for objects of the overall maxima set
- and is incrementally adapted as the single tuples x are inspected in turn.

result:

$$
\begin{aligned}
inc(x,p) = \text{if } p = 0 \\
&\text{then } x \\
&\text{else choose atom } y \le p \text{ in} \\
&\quad \text{if } x \le |a\rangle\, y \\
&\quad\quad \text{then } p \\
&\quad\quad \text{else if } y \le |a\rangle\, x \text{ then } inc(x, p-y) \\
&\quad\quad\quad\quad\quad\quad\quad\quad \text{else } y + inc(x, p-y)
\end{aligned}
$$

Conclusion

- algebraic account of an approximation relation between antichains
- induces a semilattice
- renders the maxima operator isotone in several ways
- maxima operator a closure operator in an associated preorder
- hence satisfies a Galois connection
- algebra applied to the non-trivial example of the BNL algorithm
- we are convinced that the theory will be useful for many further calculational derivations involving the maxima operator and antichains