Connections between relation algebras and cylindric algebras

Ian Hodkinson

Thanks to the RAMiCS organisers for inviting me

Introduction

Relation algebras were developed by Tarski in the early 1940s.
Relation algebras algebraise binary relations.

Cylindric algebras were also introduced by Tarski together with Chin and Thompson, in the late 1940s.
Cylindric algebras algebraise n-ary relations (for arbitrary ordinals n).

In this talk, we study some connections between them.

In particular, we outline a way of *constructing cylindric algebras from relation algebras, preserving representability both ways.*

Why?

1. Many results for cylindric algebras were proved earlier for relation algebras:

		relation algebras	cylindric algebras
1.	Non-finite axiomatisability of representable algebras	Monk 1964	Monk 1969
2.	No finite-variable axiomatisation of representable algebras	Jónsson 1988 (Tarski 1974?)	Andréka 1997
3.	No canonical axiomatisation of representable algebras	IH–Venema 2005	Bulian-IH 2013
4.	Undecidability of representability	Hirsch–IH 2001	IH 2012 $(*)$
5.	$\{$Strongly representable atom structures$\}$ non-elementary	Hirsch–IH 2002	Hirsch–IH 2009

A reasonably general way to *lift relation algebra results to cylindric algebras* would have helped.

This talk outlines a lifting that proves $(*)$.

Why?

2. There are many other algebras of relations:

- diagonal-free cylindric algebras
- polyadic algebras
- polyadic equality algebras

Reproving relation algebra results for these involves even more work!
A general lifting procedure would make it easier.

3. The relationship between relation algebras and cylindric algebras (and other kinds of algebra) is interesting in its own right.
Long history, impressive work (Monk 1961, Johnson 1969, Maddux 1978–91).
Our construction contributes to this area.

Outline of talk

1. Definitions
2. Aim of our construction
3. Earlier work
4. Outline of construction
5. Remarks, open problems

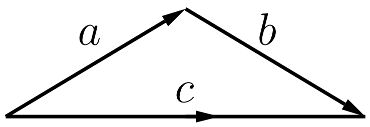
1. DEFINITIONS

A *relation algebra* is an algebra

$$\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;),$$

where

- $(A, +, -, 0, 1)$ is a boolean algebra
- $(A, ;, 1')$ is a monoid
- $\breve{}$ is a unary function on A
- $\mathcal{A}$ satisfies the *Peircean law:*
  $(a ; b) \cdot c \neq 0 \iff (\breve{a} ; c) \cdot b \neq 0 \iff (c ; \breve{b}) \cdot a \neq 0$,
  for all $a, b, c \in A$, where $a \cdot b = -(-a + -b)$.



For simplicity, in this talk we restrict to *finite simple* relation algebras $\mathcal{A}$ ($\mathcal{A}$ is *simple* iff $1 ; a ; 1 = 1$ for all $a \neq 0$).

Representations of (simple) relation algebras

A *(square) representation* of $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$ is a one-one map $h : A \to \wp(U^2)$, for some 'base' set U, such that for all $a, b \in A$,

1. $h(a+b) = h(a) \cup h(b)$ and $h(-a) = U^2 \setminus h(a)$
2. $h(0) = \emptyset$ and $h(1) = U^2$
3. $h(1') = \{(x,x) : x \in U\}$
4. $h(\breve{a}) = \{(x,y) \in U^2 : (y,x) \in h(a)\}$
5. $h(a\,;b) = \{(x,y) \in U^2 : \exists z((x,z) \in h(a) \wedge (z,y) \in h(b))\}$.

So h *represents* each $a \in A$ as a binary relation on U. The algebraic operations correspond via h to 'concrete' operations on binary relations.

Crucially, if $(x,y) \in h(a\,;b)$ then there is $z \in U$ with $(x,z) \in h(a)$ and $(z,y) \in h(b)$.
We call such a z a *witness for the composition $a\,;b$ on (x,y).*

We say that $\mathcal{A}$ is *representable* if it has a representation.

Cylindric algebras

Let $n \geq 3$. An *n-dimensional cylindric algebra* is an algebra

$$\mathcal{C} = (C, +, -, 0, 1, \mathsf{d}_{ij}, \mathsf{c}_i : i, j < n),$$

where

- $(C, +, -, 0, 1)$ is a boolean algebra as before
- the d_{ij} are constants (called *diagonals*)
- the c_i are unary functions on C (called *cylindrifiers/cylindrifications*)

$\mathcal{C}$ must satisfy certain equations (details not needed here).

In this talk we restrict to *simple* cylindric algebras:
$\mathsf{c}_0 \ldots \mathsf{c}_{n-1} a = 1$ whenever $a \neq 0$.

Idea: the elements of C are like *first-order formulas* written with variables $x_0, \ldots, x_{n-1}$.

- d_{ij} is like $x_i = x_j$
- $\mathsf{c}_i a$ is like $\exists x_i a$

Representations of (simple) cylindric algebras

Let $\mathcal{C} = (C, +, -, 0, 1, \mathsf{d}_{ij}, \mathsf{c}_i : i, j < n)$ be a (simple) n-dimensional cylindric algebra.

A *(square) representation* of $\mathcal{C}$ is a one-one map $h : C \to \wp(U^n)$, for some base set U, such that for each $i, j < n$ and each $a \in C$,

1. $h(\mathsf{d}_{ij}) = \{(x_0, \ldots, x_{n-1}) \in U^n : x_i = x_j\}$
2. $h(\mathsf{c}_i a) = \{(x_0, \ldots, x_{n-1}) \in U^n : \exists(y_0, \ldots, y_{n-1}) \in h(a)$ $(x_j = y_j$ for each $j \in n \setminus \{i\})\}$
3. h respects the boolean operations (as for relation algebras)

Each element of the algebra is 'represented' as an *n-ary relation* on U.

Again, we say that $\mathcal{C}$ is *representable* if it has a representation.

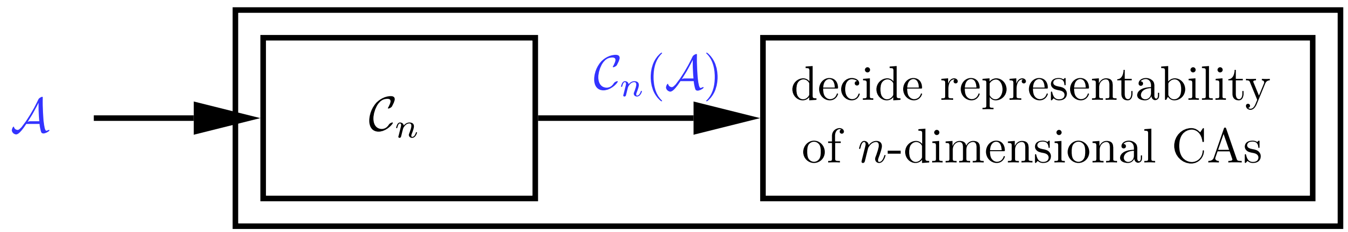
2. AIM OF OUR CONSTRUCTION

For each $n \geq 3$, we seek a *recursive reduction:* a recursive map $\mathcal{C}_n$ that

- is defined on every finite simple relation algebra $\mathcal{A}$
- takes each such $\mathcal{A}$ to a finite n-dimensional cylindric algebra $\mathcal{C}_n(\mathcal{A})$
- preserves representability both ways: for each $\mathcal{A}$,
  $\mathcal{A}$ is representable iff $\mathcal{C}_n(\mathcal{A})$ is representable

Fact 1 (Hirsch–IH 2001) *There is no algorithm to decide whether a finite simple relation algebra is representable.*

Conclusion Assuming that a recursive reduction $\mathcal{C}_n$ exists, *there is no algorithm to decide whether a finite n-dimensional cylindric algebra is representable.*



3. EARLIER WORK

J. D. Monk (Ph.D. thesis, 1961)

Monk's thesis describes a construction of a *3-dimensional* cylindric algebra from any relation algebra.

Monk states that the idea originated with Lyndon (letter to Thompson, 1949).

- The construction preserves representability both ways.
- For *finite* relation algebras, the construction is recursive.

We conclude:

Theorem 2 *There is no algorithm to decide whether a finite* 3-dimensional *cylindric algebra is representable.*

Monk gives no construction in dimensions > 3.

R. D. Maddux (Ph.D. thesis 1978, paper 1989, survey 1991)

Maddux found a method that can construct n-dimensional cylindric algebras from a finite relation algebra $\mathcal{A}$, *for arbitrary* $n \geq 3$*.*

He used objects called n*-dimensional cylindric bases.*

To explain cylindric bases, we need to introduce

- *atoms*
- *networks*

Atoms and networks

Fix a finite simple relation algebra $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$.

An *atom* of $\mathcal{A}$ is a minimal nonzero element with respect to the standard boolean ordering ($a \leq b$ iff $a + b = b$).

Write $\mathsf{At}\,\mathcal{A}$ for the set of atoms of $\mathcal{A}$.

Being finite, $\mathcal{A}$ is *atomic* — every nonzero element dominates an atom.

An *(atomic $\mathcal{A}$-)network* is a pair $N = (N_1, N_2)$, where

- N_1 is a set of 'nodes'
- $N_2 : N_1 \times N_1 \to \mathsf{At}\,\mathcal{A}$ satisfies (for all $x, y, z \in N_1$):
  1. $N_2(x, x) \leq 1'$
  2. $N_2(x, y) = N_2(y, x)\breve{}$ (this is an atom)
  3. $N_2(x, y) \leq N_2(x, z) ; N_2(z, y)$

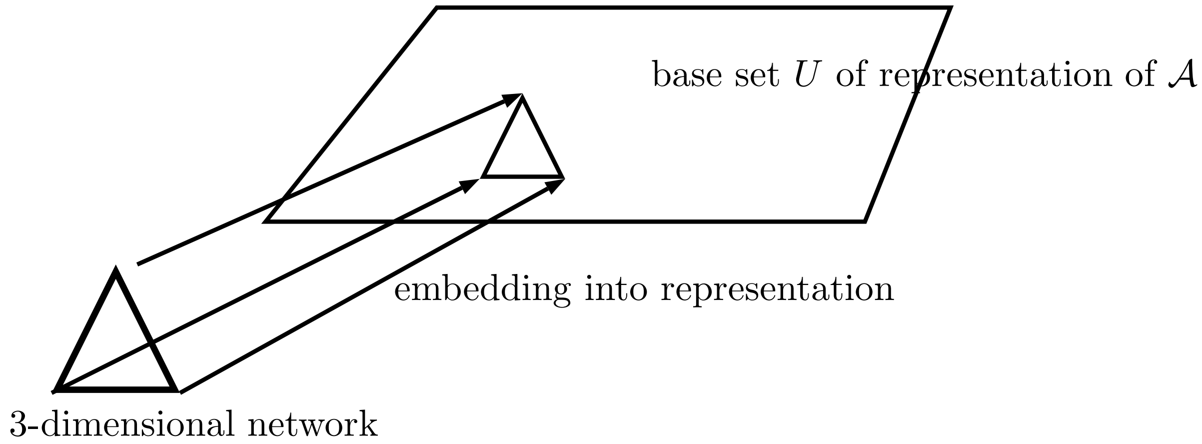
We often omit the indexes 1,2 — just write $N(x, y)$ etc.
If $n \geq 3$, we say that N *is n-dimensional* if $N_1 = n = \{0, 1, \ldots, n-1\}$.

Embedding a network into a representation

Networks can describe parts of representations.

Let $h : A \to \wp(U^2)$ be a representation of $\mathcal{A}$.
Let $N = (N_1, N_2)$ be an $\mathcal{A}$-network.

A *(partial) embedding of N into h* is a (partial) map $f : N_1 \to U$ such that $(f(x), f(y)) \in h(N_2(x, y))$ for all $x, y \in N_1$.



N describes the relations between points in $\mathsf{rng}(f)$ as specified by h.
Example: $N(x, y) \leq 1'$ iff $(f(x), f(y)) \in h(1')$ iff $f(x) = f(y)$.

Maddux's construction

An *n-dimensional cylindric basis* of $\mathcal{A}$ is a set of n-dimensional networks over $\mathcal{A}$, with certain closure properties.

Each such basis forms the set of atoms of a finite n-dimensional cylindric algebra. If this is representable, so is $\mathcal{A}$.

- The set of *all* 3-dimensional networks is a 3-dimensional cylindric basis for $\mathcal{A}$. Similarly for 4-dimensional networks.
- For $n = 3$, the resulting 3-dimensional cylindric algebra is isomorphic to Monk's — so representable iff $\mathcal{A}$ is representable.
- But for $n > 3$, the cylindric algebra may not be representable, even when $\mathcal{A}$ is representable.
- And for $n \geq 5$, $\mathcal{A}$ may not have an n-dimensional cylindric basis.

What's going on?

Embedding homogeneously

We say that N *embeds homogeneously into h* if every partial embedding of N into h extends to a total embedding.

That is, all possible *contexts* of partial embeddings of N are the same.

Fact 3 *Every 3-dimensional network embeds homogeneously into every representation of $\mathcal{A}$.*

It follows that the cylindric algebra obtained from a 3-dimensional cylindric basis is representable if $\mathcal{A}$ is representable.

But Fact 3 can fail for higher-dimensional networks.
This is 'why' the cylindric algebra got from an n-dimensional cylindric basis ($n > 3$) need not be representable, even when $\mathcal{A}$ is.

4. THE CONSTRUCTION

Fix finite $n \geq 3$. We identify n with $\{0, \ldots, n-1\}$.
We devise a new style of 'representation' of our relation algebra $\mathcal{A}$, in which clusters of $\leq n$ points yield only 3-dimensional information.

Standard notation: for a set X and integer k,

$$[X]^k = \{S \subseteq X : |S| = k\}.$$

Definition. An *n-dimensional representation* of $\mathcal{A}$ over a base set V is a family

$$\rho = \left\langle h_S : S \in [V]^{n-3} \right\rangle,$$

where:

1. Each h_S is a *loose* representation of $\mathcal{A}$ over the base set $V \setminus S$.
   'Loose' means that we don't require $h_S(1') = \{(x, x) : x \in V \setminus S\}$.

   Remark: S is a 'black hole' outside the loose representation h_S.

n-dimensional representation ctd...

2. Multiple compositions can be witnessed by a single point:

Suppose $G \subseteq V$ with $|G| < n$.

Suppose for each $S \in [G]^{n-3}$, we are given

- points $x_S, y_S \in G \setminus S$
- elements a_S, b_S of $\mathcal{A}$

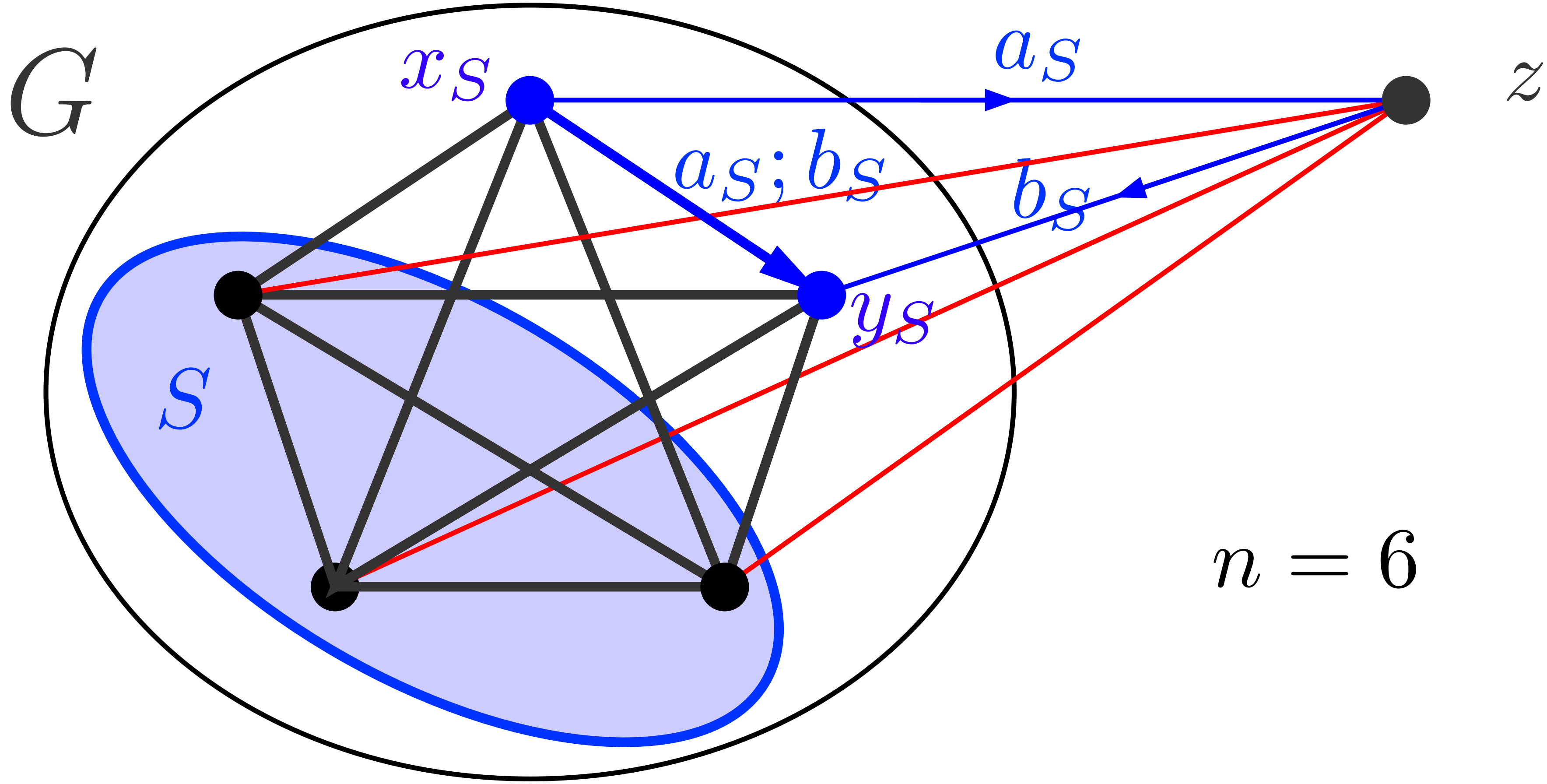
with $(x_S, y_S) \in h_S(a_S\,;b_S)$.

Suppose for each $T \in [G]^{n-4}$, we are given an $\mathcal{A}$-network N_T with nodes $G \setminus T$.

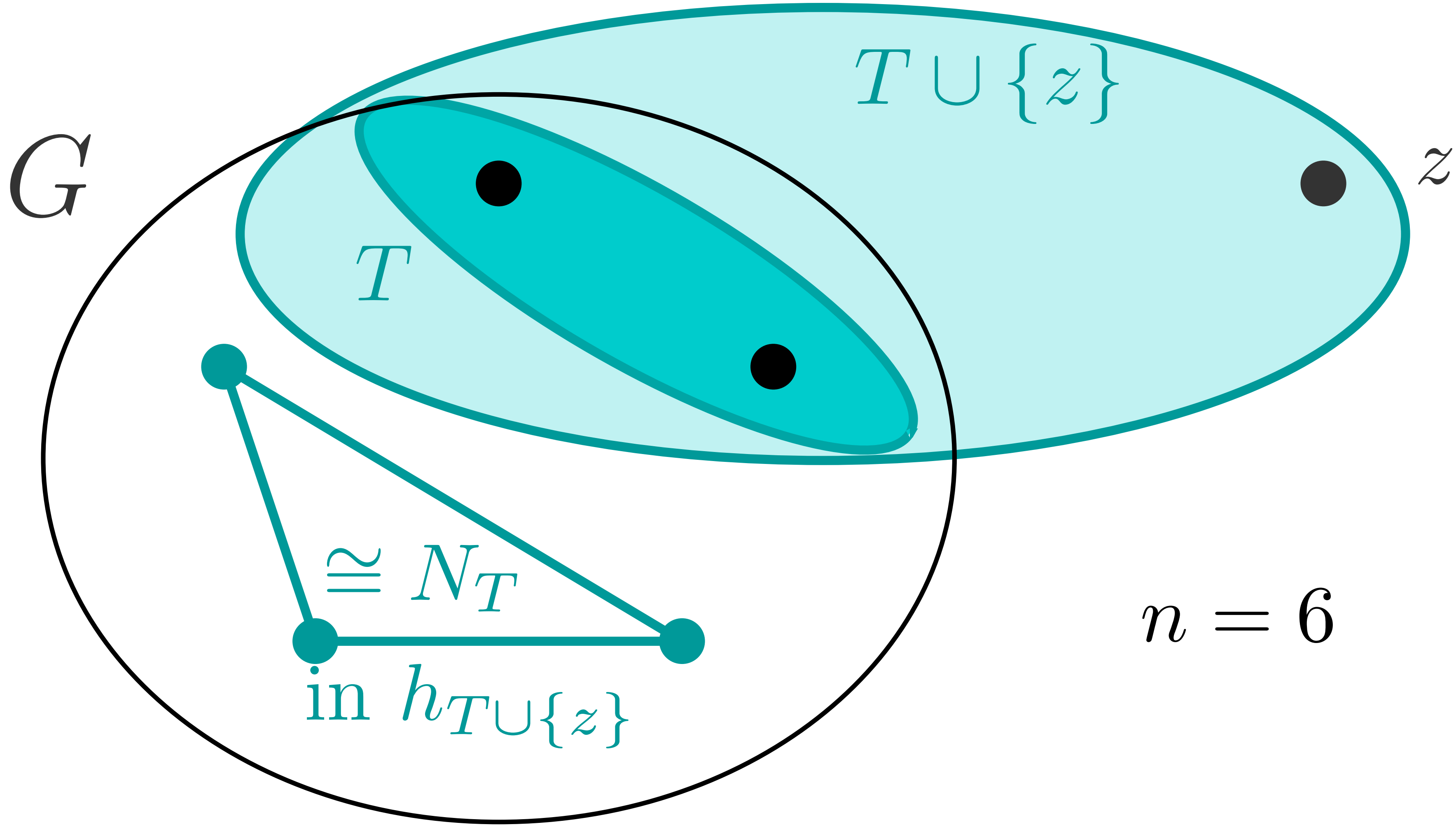
Then there is $z \in V \setminus G$ such that

1. $(x_S, z) \in h_S(a_S)$ and $(z, y_S) \in h_S(b_S)$ for each $S \in [G]^{n-3}$,
2. the identity map embeds N_T into $h_{T \cup \{z\}}$, for each $T \in [G]^{n-4}$.

Multiple compositions can be witnessed by a single point — a)



Multiple compositions can be witnessed by a single point — b)



Existence of n-dimensional representations?

Theorem 4 *$\mathcal{A}$ is representable iff it has an n-dimensional representation.*

Proof. If $\mathcal{A}$ is representable, then an n-dimensional representation can be built using a game.

If $\mathcal{A}$ has an n-dimensional representation, then $\mathcal{A}$ certainly has loose representations.
But it is easy to 'tighten' a loose representation into a conventional representation of $\mathcal{A}$. ∎

Networks for n-dimensional representations?

3-dimensional networks embed homogeneously into representations of $\mathcal{A}$. This is why Maddux's 3-dimensional cylindric bases work so well.

What kind of 'network' embeds homogeneously into an n-dimensional representation of $\mathcal{A}$?

Answer: a bundle of 3-dimensional $\mathcal{A}$-networks.
This is what we call an *(n-dimensional) hologram.*

Note: the representations h_S in an n-dimensional representation may not respect $1'$. So holograms handle equality separately, using an equivalence relation on n.

Holograms formally

— our n-dimensional analogues of 3-dimensional networks.

An *(n-dimensional) hologram (over $\mathcal{A}$)* is a family

$$\eta = \big(\sim,\ \langle N_X : X \in H(\sim)\rangle\big)$$

where

- $\sim$ is an equivalence relation on n (specifying 'logical equality')
- $H(\sim) = \{X \subseteq n : n \setminus X$ is the union of $(n-3)$ $\sim$-classes$\}$
- each N_X is an atomic $\mathcal{A}$-network whose set of nodes is X
- $N_X(i,j) \leq 1$' whenever $i, j \in X$ and $i \sim j$.

Embedding a hologram into an n-dimensional representation

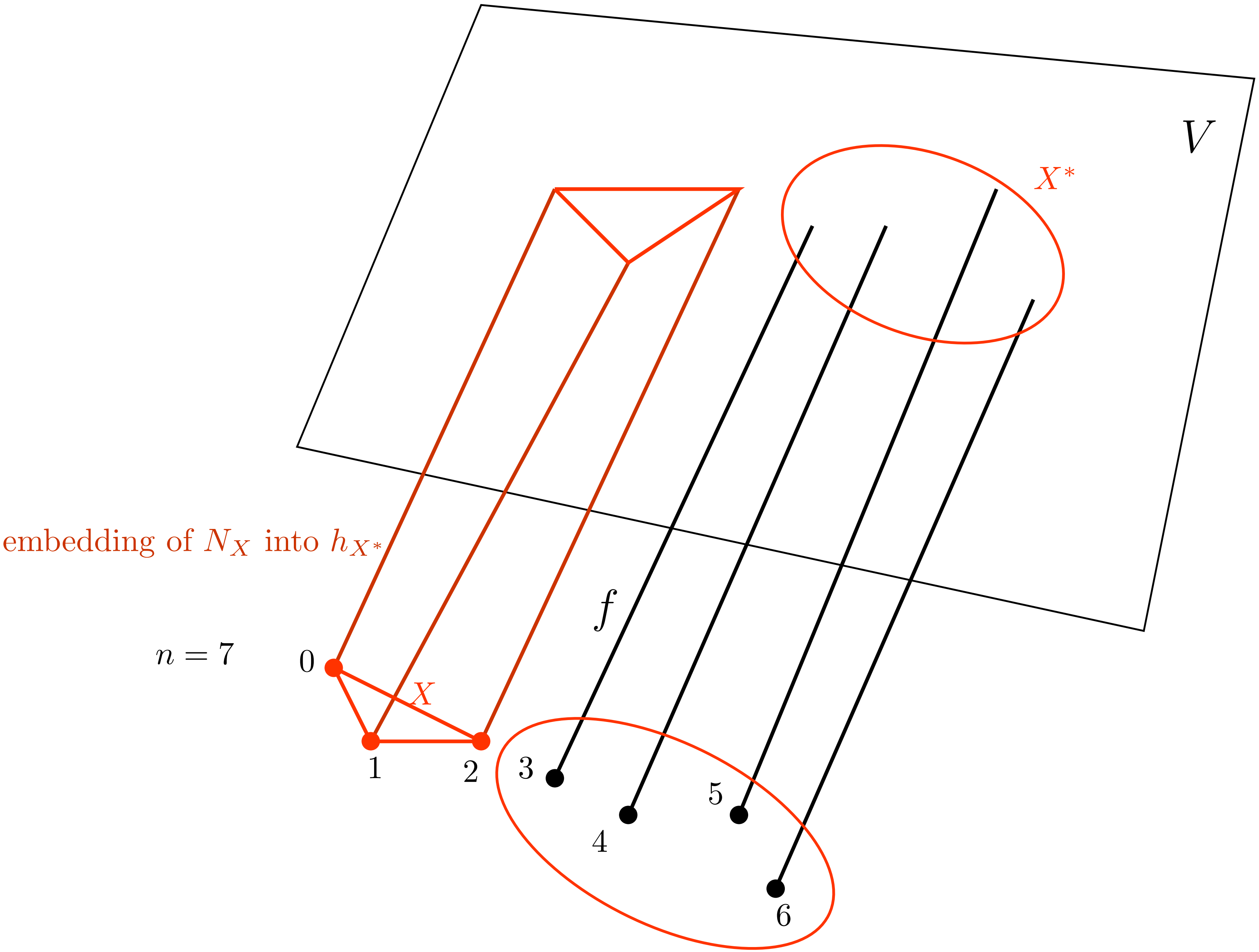
Let $\rho = \langle h_S : S \in [V]^{n-3} \rangle$ be an n-dimensional representation of $\mathcal{A}$.

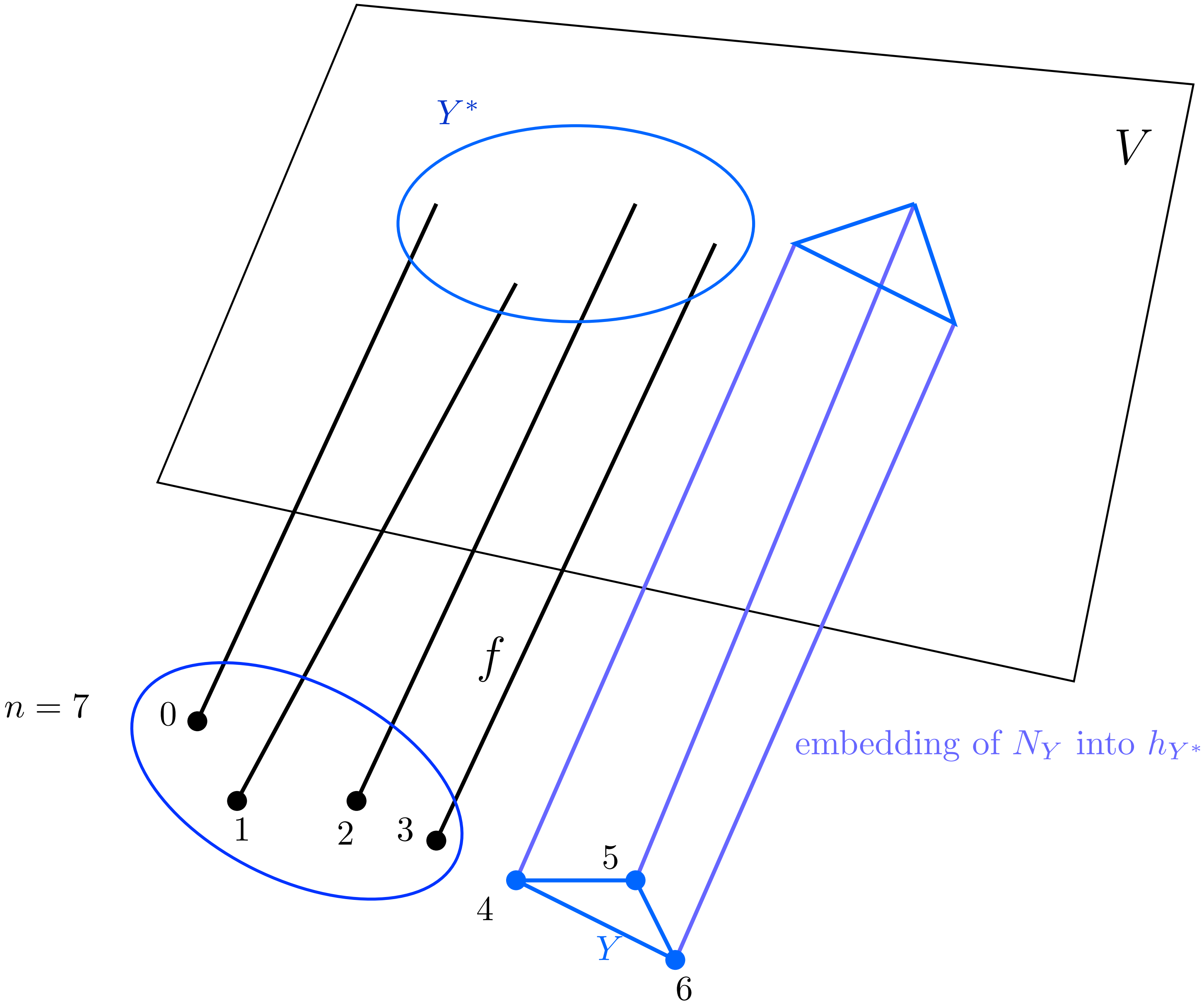
Let $\eta = (\sim, \langle N_X : X \in H(\sim) \rangle)$ be a hologram.

A *(partial) embedding of η into ρ* is a (partial) map $f : n \to V$ such that:

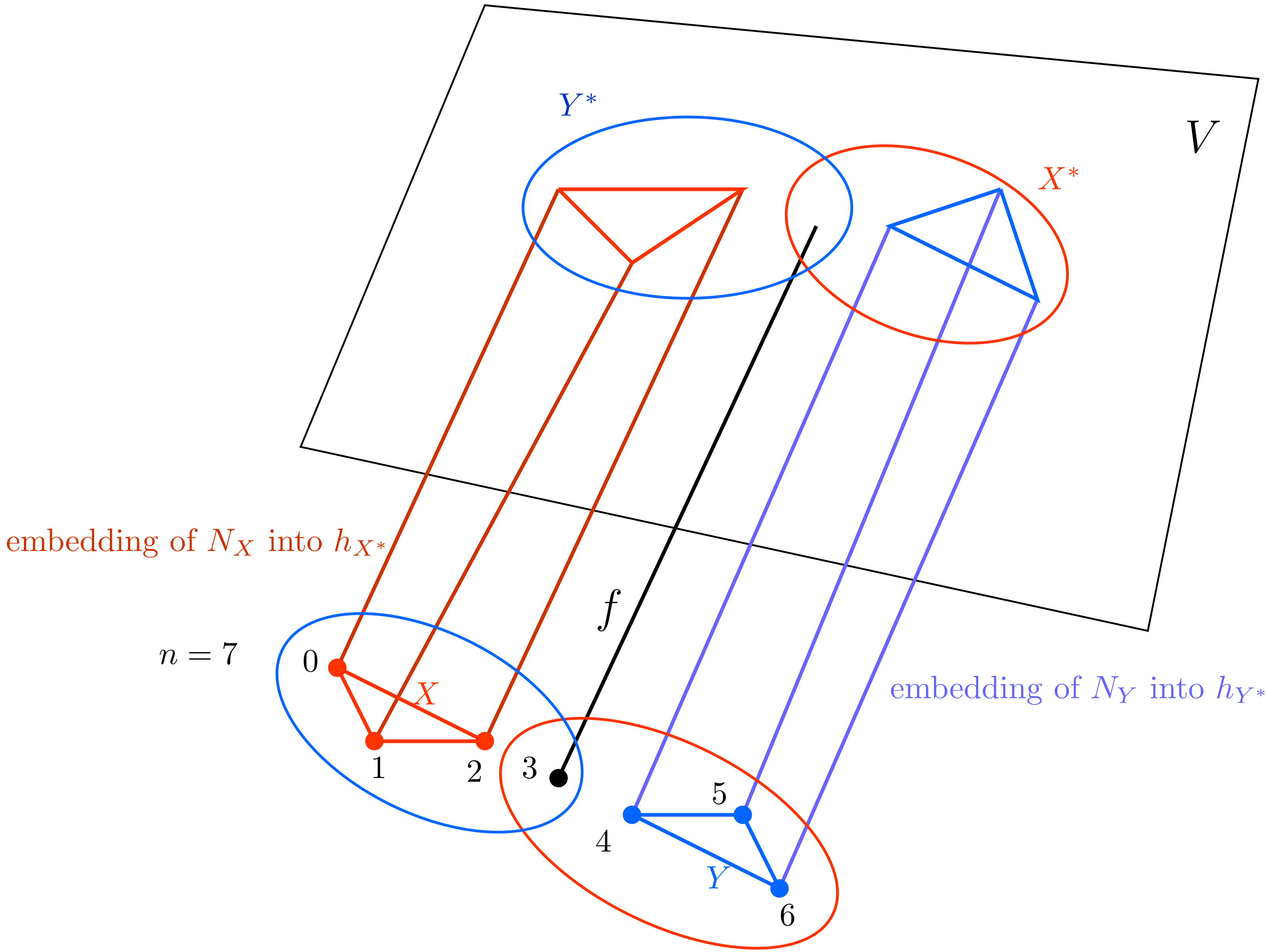
1. $f(i) = f(j)$ iff $i \sim j$, for each $i, j \in \mathrm{dom}(f)$,
2. For each $X \in H(\sim)$ with $n \setminus X \subseteq \mathrm{dom}(f)$, letting $X^* = \{f(k) : k \in n \setminus X\} \in [V]^{n-3}$, $f \restriction X$ is a (partial) embedding of the network N_X into the loose representation h_{X^*}.

We say that η *embeds homogeneously* into ρ if every partial embedding of η into ρ extends to a total embedding.


V
X*
embedding of N_X into h_{X^*}
f
n = 7
0
X
1
2
3
4
5
6



Y^*
V
f
$n = 7$
0
1
2
3
4
5
6
Y
embedding of N_Y into h_{Y^*}



Y^*
V
X^*
embedding of N_X into h_{X^*}
f
$n = 7$
0
X
1
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6
embedding of N_Y into h_{Y^*}

n-dimensional cylindric algebra from holograms

Now everything is analogous to Maddux's 3-dimensional cylindric bases.

Fact 5 *Every n-dimensional hologram embeds homogeneously into every n-dimensional representation of $\mathcal{A}$.*

Let $\mathcal{H}$ be the set of all (n-dimensional) holograms.
$\mathcal{H}$ forms the set of atoms of an n-dimensional cylindric algebra $\mathcal{C}_n(\mathcal{A})$.

Since $\mathcal{A}$ is finite, so are $\mathcal{H}$ and $\mathcal{C}_n(\mathcal{A})$.
The map $\mathcal{A} \mapsto \mathcal{C}_n(\mathcal{A})$ is recursive.

Theorem 6 *$\mathcal{C}_n(\mathcal{A})$ is representable iff $\mathcal{A}$ is representable.*

Proof. A representation of $\mathcal{C}_n(\mathcal{A})$ is essentially an n-dimensional representation of $\mathcal{A}$.
Such a representation exists iff $\mathcal{A}$ is representable (theorem 4). ■

5. CONCLUSION

Theorem 7 *For each finite $n \geq 3$, there is no algorithm to decide whether a finite n-dimensional cylindric algebra is representable.*

Proof. $\mathcal{A} \mapsto \mathcal{C}_n(\mathcal{A})$ is our desired recursive reduction. ■

Can generalise theorem 7 to

- diagonal-free cylindric algebras
- polyadic algebras
- polyadic equality algebras

Can generalise reduction to *arbitrary* simple relation algebras (not necessarily finite), by: $\mathcal{A} \mapsto \mathcal{C}_n(\mathcal{A}^\sigma)$.
But this does not seem fully satisfactory.

Might consider n-variable first-order logic interpreted over an n-dimensional representation.

Some references

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