Solving Tropical Optimization Problems via Matrix Sparsification

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Outline

Introduction: Tropical Optimization

- Tropical (idempotent) mathematics focuses on the theory and applications of semirings with idempotent addition
- The tropical optimization problems are those that are formulated and solved within the framework of tropical mathematics
- Many problems have objective functions defined on vectors over idempotent semifields (semirings with multiplicative inverses)
- Both unconstrained problems and problems with constraints in the form of linear inequalities and equalities are considered
- The problems find applications in various areas, including
  - project scheduling,
  - location analysis,
  - transportation networks,
  - decision making,
  - discrete event systems

Idempotent Algebra: Definitions and Notation

Idempotent Semifield

- *Definition:* the algebraic system $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$
- *Carrier set:* $\mathbb{X}$ with neutral elements, *zero* $\mathbb{0}$ and *identity* $\mathbb{1}$
- *Associative and commutative* binary operations: $\oplus$ and $\otimes$
- Multiplication $\otimes$ is *distributive* over addition
- Addition $\oplus$ is *idempotent:* $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication is *invertible:* for each nonzero $x \in \mathbb{X}$, there exists an inverse $x^{-1} \in \mathbb{X}$ such that $x \otimes x^{-1} = \mathbb{1}$
- *Linear order:* the order $x \leq y \Longleftrightarrow x \oplus y = y$ is a total order
- *Algebraic completeness:* the equation $x^p = a$ is solvable for any $a \in \mathbb{X}$ and integer p to provide powers with rational exponents
- *Notational convention:* the multiplication signs $\otimes$ will be omitted

Semifield $\mathbb{R}_{\max,+}$ (Max-Plus Algebra)

- *Definition:* $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$
- *Carrier set:* $\mathbb{X} = \mathbb{R} \cup \{-\infty\}$; *zero and identity:* $\mathbb{0} = -\infty$, $\mathbb{1} = 0$
- *Binary operations:* $\oplus = \max$ and $\otimes = +$
- *Idempotent addition:* $x \oplus x = x$ for all x $(\max(x,x) = x)$
- *Multiplicative inverse:* for each $x \in \mathbb{R}$, there exists x^{-1} $(= -x)$
- *Power notation:* for each $x, y \in \mathbb{R}$, there is defined x^{y} $(= xy)$
- Further examples of real idempotent semifields:

$$\begin{aligned}
\mathbb{R}_{\min,+} &= \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle, \\
\mathbb{R}_{\max,\times} &= \langle \mathbb{R}_{+} \cup \{0\}, 0, 1, \max, \times \rangle, \\
\mathbb{R}_{\min,\times} &= \langle \mathbb{R}_{+} \cup \{+\infty\}, +\infty, 1, \min, \times \rangle,
\end{aligned}$$

where $\mathbb{R}_{+} = \{x \in \mathbb{R} | x > 0\}$

Vector and Matrix Algebra Over $\mathbb{X}$

- The idempotent semifield $\mathbb{X}$ is routinely extended to the idempotent systems of vectors in $\mathbb{X}^n$ and of matrices in $\mathbb{X}^{m\times n}$
- The matrix and vector operations follow the usual entry-wise formulae with $\oplus$ as addition, and $\otimes$ as multiplication
- For any vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ in $\mathbb{X}^n$, and a scalar $x \in \mathbb{X}$, the vector operations follow the conventional rules

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_i = a_i \oplus b_i, \qquad \{x\boldsymbol{a}\}_i = x a_i$$

- For any matrices $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{m\times n}$, $\boldsymbol{B} = (b_{ij}) \in \mathbb{X}^{m\times n}$ and $\boldsymbol{C} = (c_{ij}) \in \mathbb{X}^{n\times l}$, and $x \in \mathbb{X}$, the matrix operations are given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} c_{kj}, \quad \{x\boldsymbol{A}\}_{ij} = x a_{ij}$$

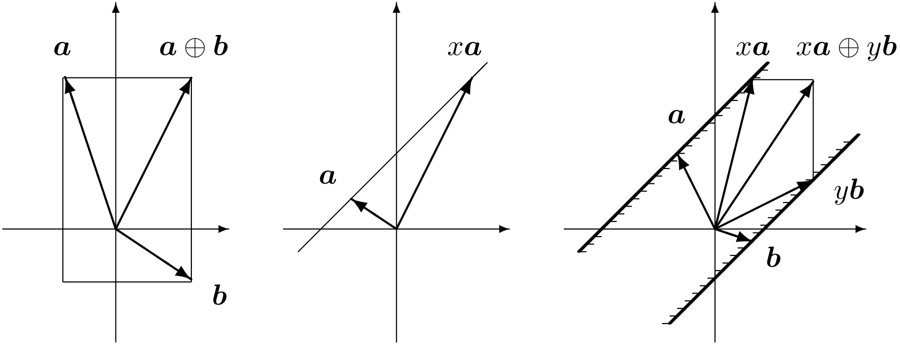
Idempotent Semimodule Over $\mathbb{X}$

- *Definition:* the system $\langle \mathbb{X}^n, \mathbf{0}, \oplus \rangle$ with scalar multiplication $\otimes$
- *Carrier set:* the set of column vectors of order n denoted $\mathbb{X}^n$
- *Zero vector:* $\mathbf{0} = (\mathbb{0}, \ldots, \mathbb{0})^T$, *vector of ones:* $\mathbf{1} = (\mathbb{1}, \ldots, \mathbb{1})^T$
- *Operations:* vector addition $\otimes$, and scalar multiplication $\otimes$
- *Regular vector:* any vector without zero components
- *Multiplicative conjugate transposition* transforms any nonzero column vector $\boldsymbol{x} = (x_i)$ into the row vector $\boldsymbol{x}^- = (x_i^-)$, where

$$x_i^- = \begin{cases} x_i^{-1}, & \text{if } x_i \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

- *Linear dependence:* a vector $\boldsymbol{y}$ is linearly dependent on vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ if $\boldsymbol{y} = c_1 \boldsymbol{x}_1 \oplus \cdots \oplus c_m \boldsymbol{x}_m$ for some scalars $c_1, \ldots, c_m$

Graphical Representation for $\mathbb{R}^2_{\max,+}$



- Addition (left), scalar multiplication (middle), and a linear span (right) of vectors in the Cartesian coordinate system in the plane

Matrices Over $\mathbb{X}$ (Further Definitions)

- *Zero* and *identity matrices:*

$$\mathbf{0} = \begin{pmatrix} \mathbb{0} & \dots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \dots & \mathbb{0} \end{pmatrix}, \qquad \boldsymbol{I} = \begin{pmatrix} \mathbb{1} & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & \mathbb{1} \end{pmatrix}$$

- *Row- (column-) regular matrix:* any matrix without rows (columns) that consist entirely of zeros
- *Multiplicative conjugate transposition* transforms any nonzero matrix $\boldsymbol{A} = (a_{ij})$ into the matrix $\boldsymbol{A}^{-} = (a_{ij}^{-})$, where

$$a_{ij}^{-} = \begin{cases} a_{ji}^{-1}, & \text{if } a_{ji} \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

Tropical Optimization Problems: Examples

Linear Objective Functions

- Hoffman (1963), Superville (1978), U. Zimmermann (1981)

$$\begin{aligned} &\text{minimize} && \boldsymbol{p}^T\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{d} \end{aligned}$$

(direct solution)

- K. Zimmermann (1984, 1992, 2003, 2006)

$$\begin{aligned} &\text{minimize} && \boldsymbol{p}^T\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}, \quad \boldsymbol{C}\boldsymbol{x} \geq \boldsymbol{b}, \\ &&& \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

(algorithmic solution)

- Butkovič (1984, 2010), Butkovič and Aminu (2009)

$$\begin{aligned} &\text{minimize} && \boldsymbol{p}^T\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d} \end{aligned}$$

(algorithmic solution)

Nonlinear Objective Functions

- Cuninghame-Green (1962, 1979), Engel and Schneider (1975), Elsner and van den Driessche (2004, 2010), K. (2013,2014)

minimize $\boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$ *(direct solution)*

- Cuninghame-Green (1976), U. Zimmermann (1981)

minimize $(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{d}$,
subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}$ *(direct solution)*

- K. Zimmermann (1984)

minimize $(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \oplus \boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}$,
subject to $\boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}$ *(algorithmic solution)*

Nonlinear Objective Functions

- K. (2004, 2009, 2012)

  minimize $(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{A}\boldsymbol{x}$ *(direct solution)*

- Butkovič and Tam (2009)

  minimize $\mathbf{1}^{T}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1}$; *(direct solution)*

  maximize $\mathbf{1}^{T}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1}$ *(direct solution)*

- Gaubert, Katz and Sergeev (2012)

  $$\begin{aligned} &\text{minimize} && (\boldsymbol{p}^{T}\boldsymbol{x} \oplus r)(\boldsymbol{q}^{T}\boldsymbol{x} \oplus s)^{-1}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d} \end{aligned}$$

  (algorithmic solution)

Solution via Matrix Sparsification: Problem Formulation

Problem

Given a matrix $\boldsymbol{A} \in \mathbb{X}^{m \times n}$ and vectors $\boldsymbol{p} \in \mathbb{X}^{m}$, $\boldsymbol{q} \in \mathbb{X}^{n}$, the problem is to find regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that

$$\text{minimize} \quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$$

- The problem appears in approximation in the sense of span seminorm (the maximum deviation between elements of a vector)
- Applications include project scheduling, decision making

Partial Solution

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Proposition

Let $\boldsymbol{A}$ be a row-regular matrix, $\boldsymbol{p}$ be nonzero and $\boldsymbol{q}$ regular vectors.

Then, the minimum is $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$, and attained at $\boldsymbol{x} = \alpha\boldsymbol{q}$ for all $\alpha > \mathbb{0}$

Sketch of Proof.

1. $\boldsymbol{x}\boldsymbol{x}^{-} \geq \boldsymbol{I} \implies (\boldsymbol{q}^{-}\boldsymbol{x})^{-1}\boldsymbol{x} = (\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-})^{-} \leq \boldsymbol{q}$
2. $(\boldsymbol{q}^{-}\boldsymbol{x})^{-1}\boldsymbol{x} \leq \boldsymbol{q} \implies (\boldsymbol{q}^{-}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{A}\boldsymbol{q}$
3. $(\boldsymbol{q}^{-}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{A}\boldsymbol{q} \implies \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \geq (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p} = \Delta$
4. $\boldsymbol{x} = \alpha\boldsymbol{q} \implies \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p} = \Delta$ □

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

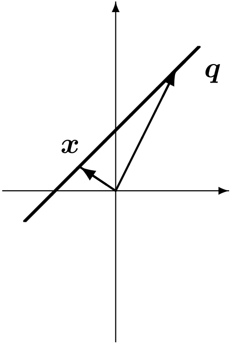
Example (in terms of $\mathbb{R}_{\max,+}$)

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Partial Solution

$$\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p} = 2,$$

$$\boldsymbol{x} = \alpha\boldsymbol{q}, \quad \alpha \in \mathbb{R}$$



Characterization of Solution

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Lemma

Let $\boldsymbol{A}$ be row-regular, $\boldsymbol{p}$ be nonzero, $\boldsymbol{q}$ be regular, and $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$.

Then, all regular solutions are given by

$$\boldsymbol{q}^{-}\boldsymbol{x} = \alpha, \quad \boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p}, \quad \alpha > \mathbb{0}$$

Sketch of Proof.

1. $\boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} = \Delta \iff \boldsymbol{q}^{-}\boldsymbol{x} = \alpha\,,\ (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} = \alpha^{-1}\Delta\,,\ \alpha > \mathbb{0}$
2. $(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} = \alpha^{-1}\Delta \iff (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \leq \alpha^{-1}\Delta\,,\ (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \geq \alpha^{-1}\Delta$
3. $\boldsymbol{q}^{-}\boldsymbol{x} = \alpha \implies \boldsymbol{x} \leq \alpha\boldsymbol{q} \implies (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \geq \alpha^{-1}\Delta$
4. $(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \leq \alpha^{-1}\Delta \iff \boldsymbol{p} \leq \alpha^{-1}\Delta\boldsymbol{A}\boldsymbol{x} \iff \boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p}$ □

Properties of Solution

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Corollary

Let $\boldsymbol{A}$ be row-regular, $\boldsymbol{p}$ be nonzero, $\boldsymbol{q}$ be regular, and $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$.

Then, the set of regular solutions is closed under addition and scalar multiplication

Sketch of Proof.
Addition:

$$\begin{aligned} \boldsymbol{q}^{-}\boldsymbol{x} = \alpha, \quad \boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p}, \\ \boldsymbol{q}^{-}\boldsymbol{y} = \beta, \quad \boldsymbol{A}\boldsymbol{y} \geq \beta\Delta^{-1}\boldsymbol{p} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \boldsymbol{q}^{-}(\boldsymbol{x} \oplus \boldsymbol{y}) = \alpha \oplus \beta, \\ \boldsymbol{A}(\boldsymbol{x} \oplus \boldsymbol{y}) \geq (\alpha \oplus \beta)\Delta^{-1}\boldsymbol{p} \end{aligned}$$

Scalar multiplication: analogously □

Matrix Sparsification

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Lemma

Let $\boldsymbol{A} = (a_{ij})$ *be a row-regular matrix,* $\boldsymbol{p} = (p_i)$ *be a nonzero vector,* $\boldsymbol{q} = (q_j)$ *be a regular vector, and* $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$ *.*

Then, replacing the matrix $\boldsymbol{A}$ *by the sparsified matrix* $\widehat{\boldsymbol{A}} = (\widehat{a}_{ij})$ *, where*

$$\widehat{a}_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} \geq \Delta^{-1} p_i q_j^{-1}; \\ \mathbb{0}, & \text{otherwise}; \end{cases}$$

does not change the solution

Sketch of Proof.

The representation $\boldsymbol{q}^{-}\boldsymbol{x} = \alpha$, $\boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p}$ yields that there may be terms $a_{ij}x_j$, which do not affect the left-hand side of the inequality □

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Example (in terms of $\mathbb{R}_{\max,+}$)

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Threshold and Sparsified Matrices

$$\Delta = 2, \qquad \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, \qquad \widehat{\boldsymbol{A}} = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & 1 \end{pmatrix}$$

Extended Solution

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Lemma

Let $\boldsymbol{A}$ be sparsified row-regular, $\boldsymbol{p}$ be nonzero, $\boldsymbol{q}$ be regular, and $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$.

Then, any vector given by the condition

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}^{-}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u}, \quad \boldsymbol{u} > \boldsymbol{0}$$

is a solution of the problem

Sketch of Proof.

1. $\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}^{-}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u} \iff \alpha\Delta^{-1}\boldsymbol{A}^{-}\boldsymbol{p} \leq \boldsymbol{x} \leq \alpha\boldsymbol{q}$
2. $\alpha\Delta^{-1}\boldsymbol{A}^{-}\boldsymbol{p} \leq \boldsymbol{x} \leq \alpha\boldsymbol{q} \implies \boldsymbol{q}^{-}\boldsymbol{x} = \alpha, \ \boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p}$ □

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Example (in terms of $\mathbb{R}_{\max,+}$)

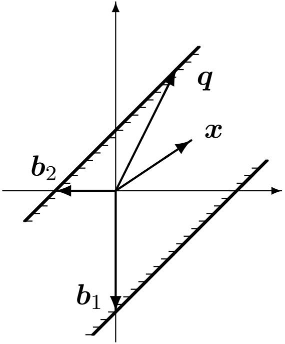
$$\boldsymbol{A} = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Extended Solution

$$\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{R}^2,$$

$$\boldsymbol{B} = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}^{-}\boldsymbol{p}\boldsymbol{q}^{-}$$

$$= \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{pmatrix}$$



All Solutions

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Theorem

Let $\boldsymbol{A}$ be sparsified row-regular, $\boldsymbol{p}$ be nonzero, $\boldsymbol{q}$ be regular, and $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$.

Let $\mathcal{A}$ be the set of matrices obtained from $\boldsymbol{A}$ by fixing one nonzero entry in each row and setting the others to $\mathbb{0}$.

Then, all regular solutions are given by

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_1^{-}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u}, \quad \boldsymbol{u} > \boldsymbol{0}, \quad \boldsymbol{A}_1 \in \mathcal{A}$$

Sketch of Proof.

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_1^{-}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u}\,, \ \boldsymbol{A}_1 \in \mathcal{A} \iff \boldsymbol{q}^{-}\boldsymbol{x} = \alpha\,, \ \boldsymbol{A}\boldsymbol{x} \geq \alpha\Delta^{-1}\boldsymbol{p} \qquad \square$$

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Example (in terms of $\mathbb{R}_{\max,+}$)

$$\boldsymbol{A} = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

All Solutions

$$\Delta = 2, \qquad \mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2\}, \qquad \boldsymbol{A}_1 = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & \mathbb{0} \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 2 & \mathbb{0} \\ \mathbb{0} & 1 \end{pmatrix};$$

$$\boldsymbol{x} = \boldsymbol{B}_1\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^2, \qquad \boldsymbol{B}_1 = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_1^{-}\boldsymbol{p}\boldsymbol{q}^{-} = \begin{pmatrix} 0 & -1 \\ \mathbb{0} & 0 \end{pmatrix};$$

$$\boldsymbol{x} = \boldsymbol{B}_2\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^2, \qquad \boldsymbol{B}_2 = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_2^{-}\boldsymbol{p}\boldsymbol{q}^{-} = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$$

Problem

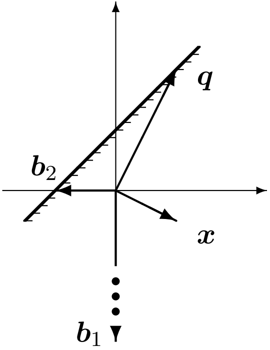
minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

All Solutions

$$\boldsymbol{x} = \boldsymbol{B}_1\boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{R}^2,$$
$$\boldsymbol{B}_1 = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_1^{-}\boldsymbol{p}\boldsymbol{q}^{-}$$
$$= \begin{pmatrix} 0 & -1 \\ \mathbb{0} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{pmatrix}$$

Example (in terms of $\mathbb{R}_{\max,+}$)

$$\boldsymbol{A} = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Complete Solution

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Theorem

Let $\boldsymbol{A}$ be sparsified row-regular, $\boldsymbol{p}$ be nonzero, $\boldsymbol{q}$ be regular, and $\Delta = (\boldsymbol{A}\boldsymbol{q})^{-}\boldsymbol{p}$.

Let $\mathcal{A}$ be the set of matrices obtained from $\boldsymbol{A}$ by leaving one nonzero entry in each row.

Let $\boldsymbol{B}_0$ be the matrix whose columns form the maximal independent system of columns in matrices $\boldsymbol{B}_1 = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{A}_1^{-}\boldsymbol{p}\boldsymbol{q}^{-}$ for all $\boldsymbol{A}_1 \in \mathcal{A}$.

Then, all regular solutions are given by $\boldsymbol{x} = \boldsymbol{B}_0\boldsymbol{u}$, $\boldsymbol{u} > \boldsymbol{0}$

Sketch of Proof.

Follows from that the set of solutions to the problem is closed under vector addition and scalar multiplication □

Problem

minimize $\quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$

Example (in terms of $\mathbb{R}_{\max,+}$)

$$\boldsymbol{A} = \begin{pmatrix} 2 & \mathbb{0} \\ 4 & 1 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Complete Solution

$$\boldsymbol{B}_1 = \begin{pmatrix} 0 & -1 \\ \mathbb{0} & 0 \end{pmatrix}, \qquad \boldsymbol{B}_2 = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix};$$

$$\boldsymbol{b}_1 = \begin{pmatrix} 0 \\ \mathbb{0} \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad \boldsymbol{b}_3 = \boldsymbol{b}_1 \oplus (-2)\boldsymbol{b}_2;$$

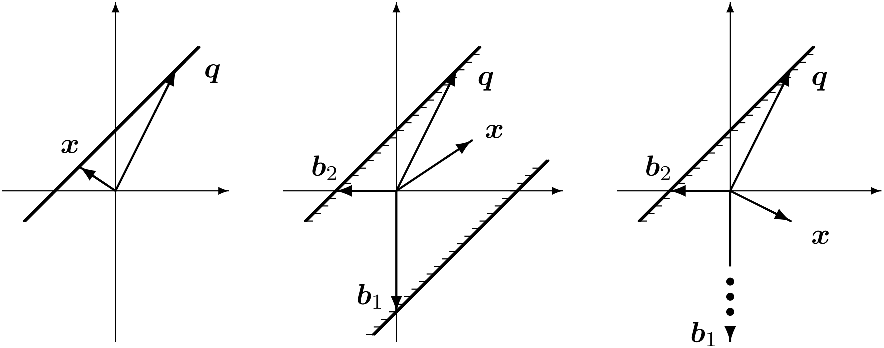
$$\boldsymbol{x} = \begin{pmatrix} 0 & -1 \\ \mathbb{0} & 0 \end{pmatrix}\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^2, \qquad \boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{pmatrix} 0 & -1 \\ \mathbb{0} & 0 \end{pmatrix}$$

Problem

minimize $\quad q^{-}x(Ax)^{-}p$

Example (in terms of $\mathbb{R}_{\max,+}$)

$$A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Concluding Remarks

- An optimization problem, which arises in approximation in the span seminorm, was examined in terms of tropical mathematics
- The problem is to minimize a function defined on vectors over an idempotent semifield by using conjugate transposition
- All solutions were characterized by simultaneous equation and inequality, and properties of the solution set were investigated
- A matrix sparsification technique was developed to derive a complete solution as a family of solution subsets
- The characteristic properties of solutions were exploited to describe the complete solution in a compact vector form
- The proposed solution approach can serve as a template to derive complete, direct solutions of other problems
- More solutions to tropical optimization problems with applications are available at **http://arxiv.org/a/krivulin_n_1**