The background of the slide is a faded, light-colored photograph of the Portico at University College London. It shows a wide stone staircase leading up to a grand building with two prominent towers in the distance, flanked by lush green trees.

# Completeness via canonicity for distributive substructural logics: a coalgebraic perspective

Fredrik Dahlqvist, David Pym  
University College London

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1 October 2015

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- A binary relation  $R$  on a poset  $(X, \leq)$  is *convex* if  $wRx$ ,  $wRz$  and  $x \leq y \leq z$  implies  $wRy$ .

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- That is the destination ... but what matters is the journey ...

## The journey: coalgebraic completeness-via-canonicity.

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Languages, Logics and free  $L$ -algebras

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- For DLC,  $L_{DLC} : \mathbf{DL} \rightarrow \mathbf{DL}$  given by

$$L_{DLC}A = F\{I, a * b, a \setminus b, a / b \mid a, b \in A\} / \\ \{(a \vee b) * c = (a * c) \vee (b * c), a * (b \vee c) = (a * b) \vee (a * c), \\ (a \vee b) \setminus c = (a \rightarrow c) \wedge (b \setminus c), a \setminus (b \wedge c) = (a \setminus b) \wedge (a \setminus c), \\ (a \wedge b) / c = (a / c) \wedge (b / c), a / (b \vee c) = (a / b) \wedge (a / c)\}$$

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- Enforcing additional axioms on  $\mathcal{L}_{IL}$  or  $\mathcal{L}_{DLC}$  = taking a (regular) quotient of  $\mathcal{L}_{IL}$  or  $\mathcal{L}_{DLC}$  = Lindenbaum-Tarski construction.

Languages, Logics and free  $L$ -algebras

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- 1951: Jónsson, Tarski define *Canonical Extensions* of BAOs
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- For any  $f : (UA)^n \rightarrow UA$ , they define  $f^\sigma : (UA^\sigma)^n \rightarrow UA^\sigma$

$$f^\sigma(x) = \bigvee \left\{ \bigwedge f[d, u] \mid K^n \ni d \leq x \leq u \in O^n \right\}$$

where  $f[d, u] = \{f(a) \mid a \in A^n, d \leq a \leq u\}$

## Canonical Extensions

### Theorem

- 1 *If  $f$  preserve binary joins in its  $i$ th argument, then  $f^\sigma$  preserves all non-empty joins in its  $i$ th argument.*
- 2 *If  $f$  preserve binary meets in its  $i$ th argument, then  $f^\sigma$  preserves all non-empty meets in its  $i$ th argument.*
- 3 *If  $f$  anti-preserve binary joins in its  $i$ th argument, then  $f^\sigma$  anti-preserves all non-empty joins in its  $i$ th argument.*
- 4 *If  $f$  anti-preserve binary meets in its  $i$ th argument, then  $f^\sigma$  anti-preserves all non-empty meets in its  $i$ th argument.*

### Corollary

*The canonical extension of an  $L_{\text{DLC}}$ -algebra is an  $L_{\text{DLC}}$ -algebra.*

## Why are canonical extensions interesting?

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### Theorem

The 'missing' axioms for IL

$$a \rightarrow a = \top \quad a \wedge (a \rightarrow b) = a \wedge b \quad (a \rightarrow b) \wedge b = b$$

and for DLC

$$\begin{array}{ll}
 a * I = I * a = a & I \leq a \setminus a, I \leq a / a \\
 a * (b \setminus c) \leq (a * b) \setminus c & (c / b) * a \leq c / (a * b) \\
 (a / b) * b \leq a & a * (b / a) \leq a
 \end{array}$$

are canonical.

# Canonical Extensions

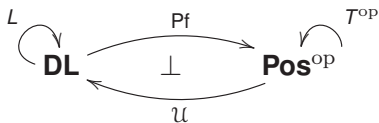
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## Jónsson-Tarski Extensions

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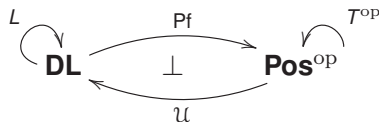
## Coalgebraic logic

- The fundamental set-up of coalgebraic logic



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- Interpret a free  $L$ -coalgebra  $\mathcal{L}$  in a  $T$ -coalgebra  $\gamma : X \rightarrow TX$  by initiality:

$$\begin{array}{ccc}
 L\mathcal{L} + FV & \xrightarrow{L[-] + \text{Id}_{FV}} & L\mathcal{U}X + FV \\
 \downarrow & & \downarrow \delta_X + \text{Id}_{FV} \\
 & & \mathcal{U}TX + FV \\
 & & \downarrow \mathcal{U}\gamma + v \\
 \mathcal{L} & \xrightarrow{[-]} & \mathcal{U}X
 \end{array}$$

## Semantics of $\mathcal{L}_{\text{IL}}$ and $\mathcal{L}_{\text{DLC}}$

- Coalgebras for  $\mathcal{L}_{\text{IL}}$ :  $T_{\text{IL}} : \mathbf{Pos} \rightarrow \mathbf{Pos}$

$$T_{\text{IL}}X = P_c(X^{\text{op}} \times X)$$

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- $\delta_X^{\text{IL}} : L_{\text{IL}}\mathcal{U}X \rightarrow \mathcal{U}T_{\text{IL}}X$

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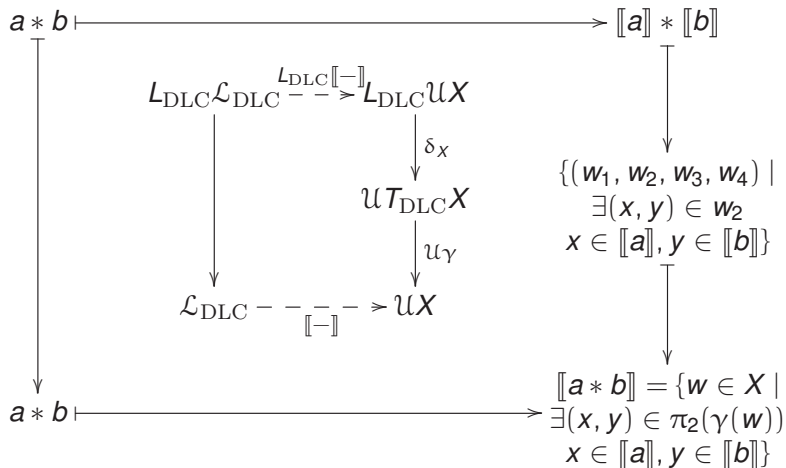
- $\delta_X^{\text{DLC}} : L_{\text{DLC}}\mathcal{U}X \rightarrow \mathcal{U}T_{\text{DLC}}X$

$$(U * V) \mapsto \{t \in T_{\text{DLC}}X \mid \exists (x, y) \in \pi_2(t), x \in U, y \in V\}$$

$$(U \setminus V) \mapsto \{t \in T_{\text{DLC}}X \mid \forall (x, y) \in \pi_3(t), x \in U \Rightarrow y \in V\}$$

$$(V/U) \mapsto \{t \in T_{\text{DLC}}X \mid \forall (x, y) \in \pi_4(t), x \in U \Rightarrow y \in V\}$$

## A worked out example



## Jónsson-Tarski Extensions

### Theorem (Coalgebraic Jónsson-Tarski theorem)

If the adjoint transpose  $\hat{\delta} = \text{Pf}L\eta \circ F\delta_F \circ \epsilon_{TF}$  has right inverses then the embedding  $\eta_A : A \rightarrow \mathcal{U}\text{Pf}A$  lifts to an  $L$ -algebra morphism as follows:

$$\begin{array}{ccccccc}
 LA & \xrightarrow{\alpha} & & & & & A \\
 \downarrow L\eta_A & & & & & & \downarrow \eta_A \\
 L\mathcal{U}\text{Pf}A & \xrightarrow{\delta_{\text{Pf}A}} & \mathcal{U}T\text{Pf}A & \xrightarrow{\mathcal{U}\hat{\delta}_A^{-1}} & \mathcal{U}\text{Pf}LA & \xrightarrow{\mathcal{U}\text{Pf}\alpha_{\text{ax}}} & \mathcal{U}\text{Pf}A
 \end{array}$$

### Theorem

The adjoint transpose of  $\delta^{\text{IL}}$  (resp.  $\delta^{\text{DLC}}$ ) has right-inverses, and  $\mathcal{L}_{\text{IL}}$  (resp.  $\mathcal{L}_{\text{DLC}}$ ) is strongly complete w.r.t.  $T_{\text{IL}}$ - (resp.  $T_{\text{DLC}}$ -) coalgebras.

## Jónsson-Tarski Extensions

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## An Isomorphism

- Using topological methods and distribution laws show that  $\alpha^\sigma$  is the unique continuous extension for a certain choice of topologies.

### Theorem

*For  $L_{\text{IL}}, T_{\text{IL}}$  and  $\delta^{\text{IL}}$  (resp.  $L_{\text{DLC}}, T_{\text{DLC}}$  and  $\delta^{\text{DLC}}$ ) the canonical extension and the Jónsson-Tarski extension are isomorphic.*

## An Isomorphism

- Using topological methods and distribution laws show that  $\alpha^\sigma$  is the unique continuous extension for a certain choice of topologies.
- Show that the structure map of the Jónsson-Tarski extension is also continuous for this choice of topologies.

### Theorem

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## Strong completeness of IL.

Let  $A_X = \{a \rightarrow a = \top, a \wedge (a \rightarrow b) = a \wedge b, (a \rightarrow b) \wedge b = b\}$ . IL is strongly complete w.r.t.  $T_{IL}$ -coalgebras on which  $A_X$  is valid.



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 & & & & & & U\text{Pf}T\mathcal{L}_{IL}/A_X \\
 & & & & & & \downarrow \\
 \mathcal{L}_{IL} & \longrightarrow & \mathcal{L}_{IL}/A_X & \xrightarrow{\eta_{\mathcal{L}_{IL}/A_X}} & U\text{Pf}\mathcal{L}_{IL}A_X & \xrightarrow{\cong} & U\text{Pf}\mathcal{L}_{IL}/A_X
 \end{array}$$

$$\Phi + A_X \not\vdash \Psi$$

$$\begin{array}{l} \Phi \not\vdash \Psi \\ \mathcal{L}_{IL}/A_X \models A_X \end{array}$$

$$\begin{array}{l} F_\Phi \cap \Psi = \emptyset \\ U\text{Pf}\mathcal{L}_{IL}/A_X \models A_X \end{array}$$

$$\begin{array}{l} \text{Pf}\mathcal{L}_{IL}/A_X \models A_X \\ F_\Phi \models \Phi \\ F_\Phi \not\models \Psi \end{array}$$

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- The technique applies much more widely, in particular it covers most logics with relational semantics.
- The method is fully modular: strong completeness for intuitionistic BI, intuitionistic ML, etc.
- We believe the technique can be applied almost unchanged to graded versions of IL, DLC, etc.



Obrigado.