

Completeness via canonicity for distributive substructural logics: a coalgebraic perspective

Fredrik Dahlqvist, David Pym
University College London

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- Result 1: Intuitionistic Logic is strongly complete w.r.t. posets with reflexive, transitive convex relations and persistent valuations.
- Result 2: Distributive Lambek Calculus is strongly complete w.r.t. posets with convex ternary relations.
- That is the destination ... but what matters is the journey ...

The journey: coalgebraic completeness-via-canonicity.

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Languages, Logics and free L -algebras

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- For IL , $L_{\text{IL}} : \mathbf{BDL} \rightarrow \mathbf{BDL}$ given by

$$L_{\text{IL}}A = F\{a \rightarrow b \mid a, b \in A\} / \{(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c), \\ a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)\}$$

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- For DLC, $L_{DLC} : \mathbf{DL} \rightarrow \mathbf{DL}$ given by

$$L_{DLC}A = F\{I, a * b, a \setminus b, a / b \mid a, b \in A\} / \\ \{(a \vee b) * c = (a * c) \vee (b * c), a * (b \vee c) = (a * b) \vee (a * c), \\ (a \vee b) \setminus c = (a \rightarrow c) \wedge (b \setminus c), a \setminus (b \wedge c) = (a \setminus b) \wedge (a \setminus c), \\ (a \wedge b) / c = (a / c) \wedge (b / c), a / (b \vee c) = (a / b) \wedge (a / c)\}$$

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- Enforcing additional axioms on \mathcal{L}_{IL} or \mathcal{L}_{DLC} = taking a (regular) quotient of \mathcal{L}_{IL} or \mathcal{L}_{DLC} = Lindenbaum-Tarski construction.

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 \mathcal{L} & \xrightarrow{q} & \mathcal{L}_{ax} & \xrightarrow{\eta_{\mathcal{L}_{ax}}} & \text{UPf}\mathcal{L}_{ax} & \xrightarrow{\cong} & \text{UPf}\mathcal{L}_{ax} \\
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Canonical Extensions

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- 1951: Jónsson, Tarski define *Canonical Extensions* of BAOs
- 1994-2004: Gehrke, Jónsson, Harding define *Canonical Extensions* of DLEs
- For any $f : (UA)^n \rightarrow UA$, they define $f^\sigma : (UA^\sigma)^n \rightarrow UA^\sigma$

$$f^\sigma(x) = \bigvee \left\{ \bigwedge f[d, u] \mid K^n \ni d \leq x \leq u \in O^n \right\}$$

where $f[d, u] = \{f(a) \mid a \in A^n, d \leq a \leq u\}$

Canonical Extensions

Theorem

- 1 *If f preserve binary joins in its i th argument, then f^σ preserves all non-empty joins in its i th argument.*
- 2 *If f preserve binary meets in its i th argument, then f^σ preserves all non-empty meets in its i th argument.*
- 3 *If f anti-preserve binary joins in its i th argument, then f^σ anti-preserves all non-empty joins in its i th argument.*
- 4 *If f anti-preserve binary meets in its i th argument, then f^σ anti-preserves all non-empty meets in its i th argument.*

Corollary

The canonical extension of an L_{DLC} -algebra is an L_{DLC} -algebra.

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Theorem

The 'missing' axioms for IL

$$a \rightarrow a = \top \quad a \wedge (a \rightarrow b) = a \wedge b \quad (a \rightarrow b) \wedge b = b$$

and for DLC

$$\begin{array}{ll}
 a * I = I * a = a & I \leq a \setminus a, I \leq a / a \\
 a * (b \setminus c) \leq (a * b) \setminus c & (c / b) * a \leq c / (a * b) \\
 (a / b) * b \leq a & a * (b / a) \leq a
 \end{array}$$

are canonical.

Canonical Extensions

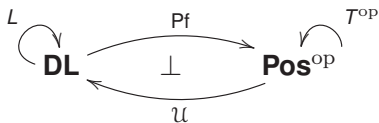
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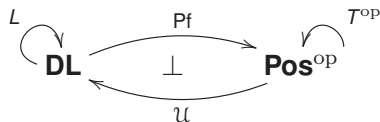
Coalgebraic logic

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- Interpret a free L -coalgebra \mathcal{L} in a T -coalgebra $\gamma : X \rightarrow TX$ by initiality:

$$\begin{array}{ccc}
 L\mathcal{L} + FV & \xrightarrow{L[-] + \text{Id}_{FV}} & L\mathcal{U}X + FV \\
 \downarrow & & \downarrow \delta_X + \text{Id}_{FV} \\
 & & \mathcal{U}TX + FV \\
 & & \downarrow u\gamma + v \\
 \mathcal{L} & \xrightarrow{[-]} & \mathcal{U}X
 \end{array}$$

Semantics of \mathcal{L}_{IL} and \mathcal{L}_{DLC}

- Coalgebras for \mathcal{L}_{IL} : $T_{\text{IL}} : \mathbf{Pos} \rightarrow \mathbf{Pos}$

$$T_{\text{IL}}X = P_c(X^{\text{op}} \times X)$$

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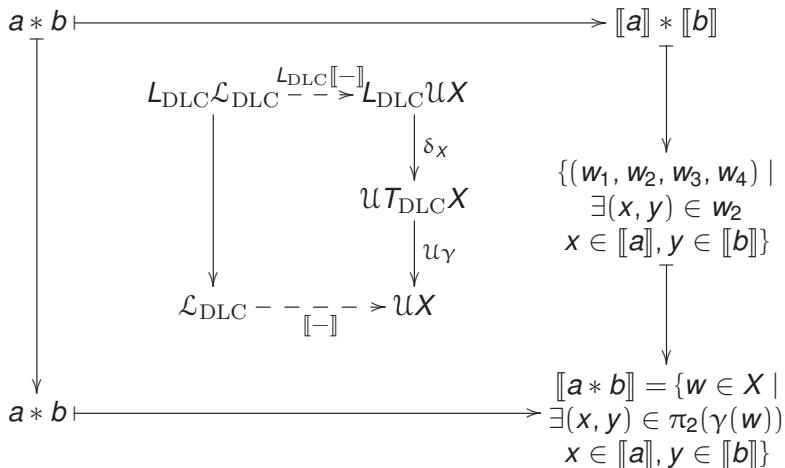
- $\delta_X^{\text{DLC}} : L_{\text{DLC}}\mathcal{U}X \rightarrow \mathcal{U}T_{\text{DLC}}X$

$$(U * V) \mapsto \{t \in T_{\text{DLC}}X \mid \exists (x, y) \in \pi_2(t), x \in U, y \in V\}$$

$$(U \setminus V) \mapsto \{t \in T_{\text{DLC}}X \mid \forall (x, y) \in \pi_3(t), x \in U \Rightarrow y \in V\}$$

$$(V/U) \mapsto \{t \in T_{\text{DLC}}X \mid \forall (x, y) \in \pi_4(t), x \in U \Rightarrow y \in V\}$$

A worked out example



Jónsson-Tarski Extensions

Theorem (Coalgebraic Jónsson-Tarski theorem)

If the adjoint transpose $\hat{\delta} = \text{Pf}L\eta \circ F\delta_F \circ \epsilon_{TF}$ has right inverses then the embedding $\eta_A : A \rightarrow \mathcal{U}\text{Pf}A$ lifts to an L -algebra morphism as follows:

$$\begin{array}{ccccccc}
 LA & \xrightarrow{\alpha} & & & & & A \\
 \downarrow L\eta_A & & & & & & \downarrow \eta_A \\
 L\mathcal{U}\text{Pf}A & \xrightarrow{\delta_{\text{Pf}A}} & \mathcal{U}T\text{Pf}A & \xrightarrow{\mathcal{U}\hat{\delta}_A^{-1}} & \mathcal{U}\text{Pf}LA & \xrightarrow{\mathcal{U}\text{Pf}\alpha_{\text{ax}}} & \mathcal{U}\text{Pf}A
 \end{array}$$

Theorem

The adjoint transpose of δ^{IL} (resp. δ^{DLC}) has right-inverses, and \mathcal{L}_{IL} (resp. \mathcal{L}_{DLC}) is strongly complete w.r.t. T_{IL} - (resp. T_{DLC} -) coalgebras.

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An Isomorphism

- Using topological methods and distribution laws show that α^σ is the unique continuous extension for a certain choice of topologies.

Theorem

For L_{IL} , T_{IL} and δ^{IL} (resp. L_{DLC} , T_{DLC} and δ^{DLC}) the canonical extension and the Jónsson-Tarski extension are isomorphic.

An Isomorphism

- Using topological methods and distribution laws show that α^σ is the unique continuous extension for a certain choice of topologies.
- Show that the structure map of the Jónsson-Tarski extension is also continuous for this choice of topologies.

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For L_{IL} , T_{IL} and δ^{IL} (resp. L_{DLC} , T_{DLC} and δ^{DLC}) the canonical extension and the Jónsson-Tarski extension are isomorphic.

Strong completeness of IL.

Let $A_X = \{a \rightarrow a = \top, a \wedge (a \rightarrow b) = a \wedge b, (a \rightarrow b) \wedge b = b\}$. IL is strongly complete w.r.t. T_{IL} -coalgebras on which A_X is valid.

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 L_{IL}\mathcal{L}_{IL} & \twoheadrightarrow & L\mathcal{L}_{IL}/A_X & \xrightarrow{L\eta_{\mathcal{L}_{IL}/A_X}} & LU\text{Pf}\mathcal{L}_{IL}/A_X & \xrightarrow{\simeq} & LU\text{Pf}\mathcal{L}_{IL}/A_X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & U\text{TPf}\mathcal{L}_{IL}/A_X \\
 & & & & & & \downarrow \\
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 & & & & & & \downarrow \\
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 \end{array}$$

$$\Phi + A_X \not\vdash \Psi$$

$$\begin{array}{l} \Phi \not\vdash \Psi \\ \mathcal{L}_{IL}/A_X \models A_X \end{array}$$

$$\begin{array}{l} F_\Phi \cap \Psi = \emptyset \\ U\text{Pf}\mathcal{L}_{IL}/A_X \models A_X \end{array}$$

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- The method is fully modular: strong completeness for intuitionistic BI, intuitionistic ML, etc.
- We believe the technique can be applied almost unchanged to graded versions of IL, DLC, etc.



Obrigado.