Completeness via canonicity for distributive substructural logics: a coalgebraic perspective

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- A binary relation R on a poset $(X, \leqslant)$ is *convex* if wRx, wRz and $x \leqslant y \leqslant z$ implies wRy.

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- Result 1: Intuitionsitic Logic is strongly complete w.r.t. posets with reflexive, transitive convex relations and persistent valuations.
- Result 2: Distributive Lambek Calculus is strongly complete w.r.t. posets with convex ternary relations.
- That is the destination ... but what matters is the journey ...

The journey: coalgebraic completeness-via-canonicity.

$$
\begin{tikzcd}
L\mathcal{L} \arrow[r, two heads, "Lq"] \arrow[ddd, "\alpha"'] & L\mathcal{L}_{\mathrm{ax}} \arrow[r, "L\eta_{\mathcal{L}_{\mathrm{ax}}}"] \arrow[ddd, "\alpha_{\mathrm{ax}}"'] & L\mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{ax}} \arrow[r, "\simeq"] \arrow[ddd, "\alpha_{\mathrm{ax}}^{\sigma}"'] & L\mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{ax}} \arrow[d, "\delta_{\mathsf{Pf}\mathcal{L}_{\mathrm{ax}}}"] \\
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Languages, Logics and free L-algebras

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- For IL, $L_{\mathrm{IL}} : \mathbf{BDL} \to \mathbf{BDL}$ given by

$$L_{\mathrm{IL}}A = \mathsf{F}\{a \to b \mid a, b \in A\}/\{(a \vee b) \to c = (a \to c) \wedge (b \to c),\\ a \to (b \wedge c) = (a \to b) \wedge (a \to c)\}$$

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- For DLC, L_{DLC} : **DL** $\to$ **DL** given by

$$L_{\mathrm{DLC}}A = \mathsf{F}\{I, a * b, a\backslash b, a/b \mid a, b \in A\}/ \\ \{(a \vee b) * c = (a * c) \vee (b * c), a * (b \vee c) = (a * b) \vee (a * c), \\ (a \vee b)\backslash c = (a \to c) \wedge (b\backslash c), a\backslash(b \wedge c) = (a\backslash b) \wedge (a\backslash c), \\ (a \wedge b)/c = (a/c) \wedge (b/c), a/(b \vee c) = (a/b) \wedge (a/c)\}$$

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- Enforcing additional axioms on $\mathcal{L}_{\mathrm{IL}}$ or $\mathcal{L}_{\mathrm{DLC}}$ = taking a (regular) quotient of $\mathcal{L}_{\mathrm{IL}}$ or $\mathcal{L}_{\mathrm{DLC}}$ = Lindenbaum-Tarski construction.

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Canonical Extensions

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- 1951: Jónsson, Tarski define *Canonical Extensions* of BAOs
- 1994-2004: Gehrke, Jónsson, Harding define *Canonical Extensions* of DLEs
- For *any* $f : (\mathrm{U}A)^n \to \mathrm{U}A$, they define $f^\sigma : (\mathrm{U}A^\sigma)^n \to \mathrm{U}A^\sigma$

$$f^\sigma(x) = \bigvee \left\{ \bigwedge f[d,u] \mid K^n \ni d \leqslant x \leqslant u \in O^n \right\}$$

where $f[d,u] = \{f(a) \mid a \in A^n, d \leqslant a \leqslant u\}$

Canonical Extensions

Theorem

1. *If f preserve binary joins in its ith argument, then f^σ preserves all non-empty joins in its ith argument.*
2. *If f preserve binary meets in its ith argument, then f^σ preserves all non-empty meets in its ith argument.*
3. *If f anti-preserve binary joins in its ith argument, then f^σ anti-preserves all non-empty joins in its ith argument.*
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Corollary

The canonical extension of an L_{DLC}-algebra is an L_{DLC}-algebra.

Why are canonical extensions interesting?

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Theorem

The 'missing' axioms for IL

$$a \to a = \top \quad a \wedge (a \to b) = a \wedge b \quad (a \to b) \wedge b = b$$

and for DLC

$$\begin{array}{ll} a * I = I * a = a & I \leqslant a\backslash a, I \leqslant a/a \\ a * (b\backslash c) \leqslant (a * b)\backslash c & (c/b) * a \leqslant c/(a * b) \\ (a/b) * b \leqslant a & a * (b/a) \leqslant a \end{array}$$

are canonical.

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Coalgebraic logic

- The fundamental set-up of coalgebraic logic

$$\begin{tikzcd} \mathbf{DL} \arrow[loop left, "L"] \arrow[rr, bend left, "\mathsf{Pf}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Pos}^{\mathrm{op}} \arrow[ll, bend left, "\mathcal{U}"] \arrow[loop right, "T^{\mathrm{op}}"] \end{tikzcd}$$

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- Interpret a free L-coalgebras $\mathcal{L}$ in a T-coalgebras $\gamma : X \to TX$ by initiality:

$$\begin{tikzcd} L\mathcal{L} + \mathsf{F}V \arrow[r, dashed, "L[\![-]\!]+\mathrm{Id}_{\mathsf{F}V}"] \arrow[dd] & L\mathcal{U}X + \mathsf{F}V \arrow[d, "\delta_X+\mathrm{Id}_{\mathsf{F}V}"] \\ & \mathcal{U}TX + \mathsf{F}V \arrow[d, "\mathcal{U}\gamma+v"] \\ \mathcal{L} \arrow[r, dashed, "[\![-]\!]"'] & \mathcal{U}X \end{tikzcd}$$

Semantics of $\mathcal{L}_{\mathrm{IL}}$ and $\mathcal{L}_{\mathrm{DLC}}$

- Coalgebras for $\mathcal{L}_{\mathrm{IL}}$: $T_{\mathrm{IL}} : \mathbf{Pos} \to \mathbf{Pos}$

$$T_{\mathrm{IL}}X = \mathsf{P}_c(X^{\mathrm{op}} \times X)$$

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- $\delta^{\mathrm{IL}}_X : L_{\mathrm{IL}} \mathcal{U} X \to \mathcal{U} T_{\mathrm{IL}} X$

$$(U \to V) \mapsto \{W \in \mathsf{P}_c(X^{\mathrm{op}} \times X) \mid \forall (x, y) \in W, x \in U \Rightarrow y \in V\}$$

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$$(U * V) \mapsto \{t \in T_{\mathrm{DLC}}X \mid \exists (x,y) \in \pi_2(t), x \in U, y \in V\}$$
$$(U \backslash V) \mapsto \{t \in T_{\mathrm{DLC}}X \mid \forall (x,y) \in \pi_3(t), x \in U \Rightarrow y \in V\}$$
$$(V / U) \mapsto \{t \in T_{\mathrm{DLC}}X \mid \forall (x,y) \in \pi_4(t), x \in U \Rightarrow y \in V\}$$

A worked out example

$$
\begin{tikzcd}
a * b \arrow[rr, mapsto] \arrow[dd, mapsto] & & {[\![a]\!] * [\![b]\!]} \arrow[d, mapsto] \\
& & {\begin{array}{c} \{(w_1, w_2, w_3, w_4) \mid \\ \exists (x, y) \in w_2 \\ x \in [\![a]\!], y \in [\![b]\!]\} \end{array}} \arrow[d, mapsto] \\
a * b \arrow[rr, mapsto] & & {\begin{array}{c} [\![a * b]\!] = \{w \in X \mid \\ \exists (x, y) \in \pi_2(\gamma(w)) \\ x \in [\![a]\!], y \in [\![b]\!]\} \end{array}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
L_{\mathrm{DLC}}\mathcal{L}_{\mathrm{DLC}} \arrow[r, dashed, "L_{\mathrm{DLC}}[\![-]\!]"] \arrow[dd] & L_{\mathrm{DLC}}\mathcal{U}X \arrow[d, "\delta_X"] \\
& \mathcal{U}T_{\mathrm{DLC}}X \arrow[d, "\mathcal{U}\gamma"] \\
\mathcal{L}_{\mathrm{DLC}} \arrow[r, dashed, "{[\![-]\!]}"'] & \mathcal{U}X
\end{tikzcd}
$$

Jónsson-Tarski Extensions

Theorem (Coalgebraic Jónsson-Tarski theorem)

If the adjoint transpose $\hat{\delta} = \mathsf{Pf}L\eta \circ F\delta_F \circ \epsilon_{TF}$ *has right inverses then the embedding* $\eta_A : A \to \mathcal{U}\mathsf{Pf}A$ *lifts to an L-algebra morphism as follows:*

$$\begin{tikzcd}
LA \arrow[rrr, "\alpha"] \arrow[d, "L\eta_A"'] & & & A \arrow[d, "\eta_A"] \\
L\mathcal{U}\mathsf{Pf}A \arrow[r, "\delta_{\mathsf{Pf}A}"'] & \mathcal{U}T\mathsf{Pf}A \arrow[r, "\mathcal{U}\hat{\delta}_A^{-1}"'] & \mathcal{U}\mathsf{Pf}LA \arrow[r, "\mathcal{U}\mathsf{Pf}\alpha_{\mathrm{ax}}"'] & \mathcal{U}\mathsf{Pf}A
\end{tikzcd}$$

Theorem

The adjoint transpose of δ^{IL} *(resp.* δ^{DLC}*) has right-inverses, and* $\mathcal{L}_{\mathrm{IL}}$ *(resp.* $\mathcal{L}_{\mathrm{DLC}}$*) is strongly complete w.r.t.* T_{IL}*- (resp.* T_{DLC}*-) coalgebras.*

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An Isomorphism

- Using topological methods and distribution laws show that α^{σ} is the unique continuous extension for a certain choice of topologies.

Theorem

For L_{IL}, T_{IL} and δ^{IL} (resp. L_{DLC}, T_{DLC} and δ^{DLC}) the canonical extension and the Jónsson-Tarski extension are isomorphic.

An Isomorphism

- Using topological methods and distribution laws show that α^σ is the unique continuous extension for a certain choice of topologies.
- Show that the structure map of the Jónsson-Tarski extension is also continuous for this choice of topologies.

Theorem

For L_{IL}, T_{IL} and δ^{IL} (resp. L_{DLC}, T_{DLC} and δ^{DLC}) the canonical extension and the Jónsson-Tarski extension are isomorphic.

Strong completeness of IL.

Let $\mathrm{Ax} = \{a \to a = \top, a \wedge (a \to b) = a \wedge b, (a \to b) \wedge b = b\}$. IL is strongly complete w.r.t. T_{IL}-coalgebras on which Ax is valid.

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$$
\begin{tikzcd}
L_{\mathrm{IL}}\mathcal{L}_{\mathrm{IL}} \arrow[r, two heads] \arrow[ddd] & L\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[r, "L\eta_{\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax}}"] \arrow[ddd] & L\mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[r, "\simeq"] \arrow[ddd] & L\mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[d] \\
 & & & \mathcal{U}T\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[d] \\
 & & & \mathcal{U}\mathsf{Pf}T\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[d] \\
\mathcal{L}_{\mathrm{IL}} \arrow[r, two heads] & \mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \arrow[r, "\eta_{\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax}}"'] & \mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}\mathrm{Ax} \arrow[r, "\simeq"'] & \mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax}
\end{tikzcd}
$$

$\Phi + \mathrm{Ax} \not\vdash \Psi$	$\Phi \not\vdash \Psi$ $\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \models \mathrm{Ax}$	$F_\Phi \cap \Psi = \emptyset$ $\mathcal{U}\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \models \mathrm{Ax}$	$\mathsf{Pf}\mathcal{L}_{\mathrm{IL}}/\mathrm{Ax} \models \mathrm{Ax}$ $F_\Phi \models \Phi$ $F_\Phi \not\models \Psi$

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Conclusion

- We have shown strong completeness of IL and DLC using coalgebraic completeness-via-canonicity.
- The technique applies much more widely, in particular it covers most logics with relational semantics.
- The method is fully modular: strong completeness for intuitionistic BI, intuitionistic ML, etc.
- We believe the technique can be applied almost unchanged to graded versions of IL, DLC, etc.

Obrigado.