On the ‘A’ that links the ‘M’s of Maths, Music and Maps

J.N. Oliveira

Dept. Informática,
Universidade do Minho
Braga, Portugal

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Someone has observed that "those who like maths also enjoy music and maps" \(^1\):

\[
\begin{array}{c}
\text{MATHS} \\
\downarrow \\
\text{MUSIC} \quad \text{MAPS}
\end{array}
\]

What commonalities can be found between such a language system (Maths), an art (music) and a science (cartography)?
Maths is an *abstract* language (A)
Music is perhaps the most *abstract* (A) of all arts
Maps are geographical *abstractions* (A)
Quoting Jeff Kramer\(^2\):

"**Abstraction** is widely used in other disciplines such as **art** and **music**. For instance (...) Henri Matisse manages to clearly represent the **essence** of his subject, a naked woman, using only simple lines or cutouts. His representation **removes** all detail yet **conveys** much."

The famous “abstract” map of the London Underground (1939)

**Base principle:**
“Eliminate characteristics of the mapped object that are **not relevant** to the map’s purpose”
Abstraction

Opus 118, no.2, by Johannes Brahms (1833-97):

Questions:

*What does this piece mean? Does it describe anything? Does it imitate or recreate reality?*

Answer:

*It means nothing — it is abstract!*

Leonard Bernstein (1st Young People’s Concerts, 18-Jan-1958):

“Music is never about anything: music just ‘is’!”
Albert Camus (1913-60):

*Music is the **perfect expression of an ideal world** which is communicated to us through harmony. This world exists. Not at a level higher or lower than the real world, but parallel to it.*

*World of ideas? Maybe. *Or else world of numbers,* as communicated to us by Harmony.*”

(Essay on Music, 1932)
Mathematics is the universal language of science. Why?

- The abstract language par excellence
- Safe (unambiguous) means to pass knowledge between generations
- Mathematical proof — the ideal way to provide verifiable evidence.

Abstraction: what is it, after all?

Our answer will be based on a concept of mathematics itself — that of a function.
Two functions

Number of letters ($nr$) and stress vowel ($sv$)

\[ V \leftarrow sv \quad W \rightarrow nr \rightarrow N \]

'e' ← "Einstein" → 8

'i' ← "picnic" → 6

'a' ← "almond"

Notation (Leibniz): $6 = nr("almond"), 'a' = sv("almond"), etc

In general:

\[ y = f(x) \]
Given some function \( A \xrightarrow{f} B \) such that, for all \( b \) in \( B \), there is some \( a \) from \( A \) such that \( b = f(a) \), for example

\[
\begin{array}{ccc}
W & \xrightarrow{sv} & V \\
"almond" & \rightarrow & 'a' \\
"Einstein" & \rightarrow & 'e' \\
"picnic" & \rightarrow & 'i' \\
"poll" & \rightarrow & 'o' \\
"future" & \rightarrow & 'u'
\end{array}
\]

we say that domain \( B \) is more abstract than \( A \) and that \( f \) is a witness of such an abstraction.

In the example: one stress Vowel abstracts many Words.
Textual abstraction functions

(a) As Albert is as smart as a mouse and fast as a hare... ... he fled without a trace
    \[ f \downarrow \]

(b) Albert ... smart ... mouse ... fast ... hare... ... fled without a trace
    \[ g \downarrow \]

(c) Mousy 'Mr. Hare'... ... fled without a trace
    \[ \]

We see that by losing textual information, the text acquires a metaphorical dimension — e.g. nickname 'Hare', and so on.

"A la Chomsky": functions \( f \) and \( g \) transform deep structure (a) into surface structure (c).
How two functions make a metaphor

Example:

\[ \text{mouse} \quad f \quad \text{smart} \quad g \quad \text{Albert} \]

where \( f \) and \( g \) are the witnesses of the metaphor.

In the example there are two juxtaposed metaphors:

\[ \text{mouse} \quad f_1 \quad \text{smart} \quad g_1 \quad \text{Albert} \quad f_2 \quad \text{fast} \quad g_2 \quad \text{hare} \]
Metaphors in music

Listen to the music excerpt

which includes two simple metaphors, one **melodic** and the other **rhythmic**:
A metaphor can be regarded as a “cospan” (aside) in which

- \( T \) (tenor) is the subject
- \( V \) is the vehicle
- \( A \) is the shared attribute.


Summing up, a metaphor is a **binary relationship**

\[ T(f \circ g) V \]  \hspace{1cm} (1)

in which the attribute (\( A \)) is hidden.
Metaphors as maths objects

Brief explanation of the formula in the previous slide:

- $R^\circ$ denotes $R$ in passive voice: $b R a$ thus means the same as $b R^\circ a$. Example:
  
  *Albert watches the hare*

  versus

  *the hare is watched by Albert*

- Composition: $(f \cdot g)x = f(gx)$, for instance

  $f \ p = \text{the number of letters of } \ p$

  $g \ p = \text{the stressed syllable of } \ p$

  Thus:

  $(f \cdot g)p = \text{the number of letters of the stressed syllable of } \ p$
Putting the **vehicle** in motion

‘No jobs for the boys’ in metaphorical form, quoting Eça de Queirós (1845-1900):³

“*Os políticos e as fraldas devem trocar-se frequentemente e pela mesma razão*”

**Metaphor:**

\[
\begin{align*}
P & \xleftarrow{c^o \cdot s} F \\
& \xrightarrow{c} B \\
& \xrightarrow{s} \\
\end{align*}
\]

“Axiom: \( s \cdot (t \cdot x) = \text{False} \) — inducing change \( t' \) in \( P \), etc etc.

³Abbreviations: \( P = \text{politician} \) (tenor); \( F = \text{nappy} \) (vehicle); \( c = \text{corrupted} \); \( s = \text{dirty} \); \( t = \text{change} \).
Metaphors and ambiguity

In the presence of more than one shared attribute, e.g.

```
   T <= f <= h <= V
   \downarrow    \downarrow    \downarrow    \downarrow
   A             g           h             B
```

attribute omission leads to **ambiguity**.

Either **context** suggests the attribute or the text becomes **open** and may acquire a **poetic** dimension.

Metaphors often “close” inter-textually — see next slide.
Metaphors in music

A well-known anecdote (Vienna, 1820s):

Vaterländischer Künstlerverein (by 51 composers) on a waltz by Anton Diabelli (1781-1858).
Var. XXII of contribution 51

A. Diabelli (1781-1858)

W.A. Mozart (1756-1791)

Don Giov.: Notte e giorno faticar (Fg, Vc, Ba)

L. van Beethoven (1770-1827)

op120.xxii

etc
Metaphors in music

Funeral march by Gustav Mahler (1860-1911): metaphor with tenor

whose vehicle is the popular:

Fre re Jac ques, Fre re Jac ques,

dor mez vous? Dor mez vous? Son nez les ma ti nes!

Son nez les ma ti nes! Din dan don. Din dan don.
(More abstract) metaphors in music

3rd mov. of Brahms (1833-97) 1st symphony:

Sentence alone is metaphoric by itself, why?
- Too obvious: two halves share the same rythm
- Less obvious: 2nd half (tenor) is an inversion of 1st half (vehicle).

Music can be very ‘metaphoric’ in this (rather formal) sense.
José Saramago (1922-2010):

Deus está em mim,

\[ A \rightarrow B \]
\[ B \rightarrow \neg A \]

ou em mim não está Deus

António Vieira (1608-97):

Os homens (tinham) a razão sem uso

\[ A \rightarrow \neg B \]
\[ B \rightarrow \neg A \]

os peixes o uso sem a razão

(shaped like the letter \( \chi \), cf. root \( \chi\alpha\sigma\mu\varsigma \) from the Greek)
Chiasmus in music

The sequence $A \ B \ ¬B \ ¬A$ of terms that cross each other in the *parallel* or *double antithesis* of a chiasmus becomes *parallel de facto* in music, thanks to *polyphony*:

Antithesis by retrograde motion (cf. *algebraic* sequence inversion) exhibiting properties such as e.g.

$$¬¬A = A$$

$$¬(A \ B) = (¬B) (¬A)$$
Chiasmus in music

Abundant device in e.g. Baroque music, cf. *Canon a 2 super theme regium* from BVW 1079 by J.S. Bach (1685-1750):

A =

\[ A = \]

\[\begin{array}{c}
\text{\( J = 200 \)} \\
\end{array}\]

B =

\[ B = \]

\[\begin{array}{c}
\text{\( J = 200 \)} \\
\end{array}\]
Musical Offering BWV 1079

Canon a 2
(nr.1 of Canones diversi super thema regium, BWV 1079)

J.S. Bach (1685-1750)
The stamp of J.S. Bach (1685-1750) in which the initials 'J', 'S', 'B' overlap each other once “mirrored” (ie. inverted), cf.
Algebraicity

Apparently a requirement “wilfully” claimed by Saussure in his writings (ELG, p. 236): 4

“l’expression simple sera algébrique ou ne sera pas”

In what measure can universal algebra help?

Music particularly interesting in this respect, for its inherent algebraicity:

- Music event level forms a very simple algebra: that of sequences of pairs (pitch, duration).
- Transformations across musical metaphor witnesses easy to express and reason about (many can be regarded as linear transforms).

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Algebraicity

Take **augmentation**, for instance:

\[
\begin{align*}
k \ast [] &= [] \\
k \ast [(p, d)] &= [(p, k \ d)] \\
k \ast (m \ n) &= (k \ast m) (k \ast n)
\end{align*}
\]

**Transposition:**

\[
\begin{align*}
[] + i &= [] \\
[(p, d)] + i &= [(p + i, d)] \\
(m \ n) + i &= (m + i) (m + i)
\end{align*}
\]

**Retrograde:**

\[
\begin{align*}
\neg[] &= [] \\
\neg[(p, d)] &= [(p, d)] \\
\neg(m \ n) &= (\neg n) (\neg m)
\end{align*}
\]

e tc
Algebraicity

In general, can one “measure” algebraically the expressive richness of a metaphor $f \circ g$?

**Algebra** of functions: $f \leq g$ measuring loss of information — $f$ abstracts more than $g$ ($\equiv f$ is less injective than $g$).

Distance between tenor and vehicle of metaphor $f \circ g$ measurable by **complements** $\neg f$ and $\neg g$ ($\equiv$ what is **not** common), where $\neg f$ satisfies the universal property

$$id \leq f \triangle k \equiv \neg f \leq k$$

where $id$ is the identity and $(f \triangle g) x = (f x, g x)$.

For example, small complements will correspond to **poor** metaphors, etc.
"He immersed himself in Chomskyan linguistics (...) so that he could then apply the principles of linguistics to music — thereby creating a brand new field of study.

Ambitious? Oh, yes! Was he in over his head? Completely!"

(Jamie Bernstein).
For Leonard Bernstein (1919-90), music is

- inherently **transformational**
- the **most metaphorical** of all means of artistic expression.

In fact:

- musical processing functions are easy to identify (cf. **algebra** of sequences);
- metaphors in music are **inherently abstract** — they close up literally on the musical text alone.

Expressive efficacy proportional to the ‘**metaphorical engineering**’ of the composer — have a look at our last example (next slides).
Creative process

The example leads us back to Chomsky’s (hidden) **deep structures** (prosaic, vulgar) as opposed to the (visible, audible) **surface structures** (poetic, beautiful).

In a sense, isn’t this **deep-to-surface** transformation an essential part of the **creative** process itself?

Sketch books of L. van Beethoven (1770-1827) — 7000 manuscript folia (many on the internet\(^5\)) ready for this kind of study.

**Deep structure** starting point\(^6\): a vulgar, ‘Mannheim-like’ theme (see next slide).

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Deep structure

From this, sonata opus 31-no.2 eventually emerged:
“Mannheim rocket” theme in the sketches is the hidden (common) attribute of the whole movement’s metaphor.

**NB:** see chap. 12 of *Cooper, op.cit.*, for a detailed study of the sketches of this movement of the sonata.
“Metaphors We Live By”

In their classic textbook, George Lakoff and Mark Johnson show how metaphorical “normal language” is in everyday life, eg.:

- **Argument is War** (read: $\text{Argument} \xleftarrow{f \circ g} \text{War}$ for some witnessing $f, g$)

- **Time is Money** (read: $\text{Time} \xleftarrow{f \circ g} \text{Money}$ for some other witnesses $f, g$)

My question is:

*Since our brain structures concepts and knowledge around so many “down-to-earth” metaphors, how do such metaphors acquire the æstetical drive which lifts us to Camus’ (parallel) ideal world?*

Such is the magic of a **creative mind**.
Towards the Semantics of Music

Mihailo Antovic\(^7\) (Univ. Nis, Serbia):

Music is an abstraction, and the only way to approach it is to metaphorise — i.e. map the concrete onto the abstract, be it through waterfalls, bamboos or dots on a vertical line.

(...) conceptual metaphor theory, in its search of the conceptualization of music, provides the most solid grounds for a true ‘musico-semantics’.

Afterthought

Two flavours in (applied) **linguistics**,

- **generative** (grammars, parsing ....)
- **cognitive** ("metaphors we live by"...)

Parallel in software science:

- “Hylomorphisms” with pattern $f \cdot g^\circ$, eg. context-free compilers:

  $$\text{compiler} = \text{code\_generator} \cdot \text{pretty\_printer}^\circ$$

- “Metaphorisms” with pattern $f^\circ \cdot g$, eg. sorting:

  $$\text{sort} = \text{is\_ordered} \cdot (\text{bagify}^\circ \cdot \text{bagify})$$

cf. $aaaabbc \xrightarrow{\text{bagify}} a4b2c1 \xleftarrow{\text{bagify}} abacaba$
Humanities versus Science

- The *Big Divide*: “Art” or “Science” ... — a historical error.
- Nefarious disjunction since the era of specialization.
- Man of Hellenism and of the Renaissance lost.
- Some *counter-examples* in Portugal: João de Freitas Branco (1922-89), Jorge de Sena (1919-78), Rómulo de Carvalho (1906-97) ...
- Sociology of mathematics — the real problem?
- Confluence requires change of attitude on both sides, with much work ahead.

Music well positioned to “bridge the gap” ...
- Experience in *Computer-aided Musicology* course at Minho.
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