Relational Formalisations of Compositions and Liftings of Multirelations

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Our contributions

- Relational formalization of 3 kinds of compositions by introducing the liftings of multirelations.

  Kleisli's composition:

  $$\alpha \circ \beta = \alpha\beta_\circ$$

  Peleg's composition:

  $$\alpha * \beta = \alpha\beta_*$$

  Parikh's composition:

  $$\alpha \diamond \beta = \alpha\beta_\diamond$$

  $\beta_\circ$: Kleisli lifting, β_* : Peleg lifting, $\beta_\diamond$: Parikh lifting

  $\beta_\circ, \beta_*, \beta_\diamond : \wp(Y) \rightharpoonup \wp(Z)$

Our contributions

- We give subclasses of multirelations that form categories with each composition, respectively.

subclass	composition	the unit
mappings $f : X \to \wp(Y)$	$\alpha \circ \beta$ (Kleisli)	the singleton map $\{(a, \{a\}) \mid a \in X\}$
union-closed multirelations	$\alpha * \beta$ (Peleg)	the singleton map $\{(a, \{a\}) \mid a \in X\}$
up-closed multirelations	$\alpha \diamond \beta$ (Parikh)	the membership rel. $\{(a, A) \mid a \in A\}$

Outline

1. Kleisli lifting and Kleisli's composition
2. Peleg lifting and Peleg's composition
3. Parikh lifting and Parikh's composition
4. Associativity and the unit of each composition

Kleisli's composition

Proposition

For $\alpha : X \rightharpoonup \wp(Y)$, $\beta : Y \rightharpoonup \wp(Z)$

$$\alpha \circ \beta = \alpha\beta_\circ$$

where $\beta_\circ$ *is the Kleisli lifting of* β*.*

We introduce the Kleisli lifting $\beta_\circ$ so that

$$(B, A) \in \beta_\circ \Leftrightarrow A = \bigcup \beta(B)$$

$$\beta(B) = \{C \mid \exists b \in B.(b, C) \in \beta\}$$

Kleisli lifting

Definition

For $\beta\colon Y \rightharpoonup \wp(Z)$, define $\beta_\circ\colon \wp(Y) \rightharpoonup \wp(Z)$ by

$$\beta_\circ = \wp(\beta \ni_Z)$$

$\ni_Z$: the converse of the membership relation

$(B, A) \in \wp(\beta \ni_Z) \Leftrightarrow a \in A \leftrightarrow \exists b \in B.(b, a) \in \beta \ni_Z$

Peleg's composition

Proposition

For $\alpha : X \rightharpoondown \wp(Y)$, $\beta : Y \rightharpoondown \wp(Z)$

$$\alpha * \beta = \alpha\beta_*$$

where β_* is the Peleg lifting of β.

We introduce the Peleg lifting β_* so that

$$(B, A) \in \beta_* \Leftrightarrow \exists f. (\forall b \in B. (b, f(b)) \in \beta) \wedge A = \bigcup f(B)$$

$$f(B) = \{C \mid \exists b \in B.(b, C) \in f\}$$

Peleg lifting

Definition

For $\beta\colon Y \rightharpoonup \wp(Z)$, define $\beta_*\colon \wp(Y) \rightharpoonup \wp(Z)$ by

$$\beta_* = \bigsqcup_{f \sqsubseteq_c \beta} \hat{u}_{\lfloor\beta\rfloor} f_\circ$$

$f_\circ$: the Kleisli lifting of f

$\lfloor\beta\rfloor$: the relational domain of β

$f \sqsubseteq_c \beta \Leftrightarrow f \sqsubseteq \beta \wedge f$: pfn $\wedge \lfloor f \rfloor = \lfloor\beta\rfloor$

$\hat{u}_{\lfloor\beta\rfloor}$: the power subidentity of $\lfloor\beta\rfloor$

The *power subidentity* $\hat{u}_v \sqsubseteq \mathrm{id}_{\wp(Y)}$ of $v \sqsubseteq \mathrm{id}_Y$ is defined by

$$(A, A) \in \hat{u}_v \;\Leftrightarrow\; \forall a \in A.\ (a, a) \in v$$

Parikh's composition

Proposition

For $\alpha : X \rightharpoonup \wp(Y)$, $\beta : Y \rightharpoonup \wp(Z)$

$$\alpha \diamond \beta = \alpha\beta_\diamond$$

where $\beta_\diamond$ *is the Parikh lifting of* β*.*

We introduce the Parikh lifting $\beta_\diamond$ so that

$$(B, A) \in \beta_\diamond \iff \forall b \in B.\ (b, A) \in \beta$$

Parikh lifting

Definition

For $\beta\colon Y \rightharpoonup \wp(Z)$, we define $\beta_\diamond\colon \wp(Y) \rightharpoonup \wp(Z)$ by

$$\beta_\diamond = \ni_Y \rhd \beta$$

$\rhd$: the right residuation

$$(B, A) \in \ni_Y \rhd \beta \;\Leftrightarrow\; \forall y \in Y.\, (\,(B, b) \in \ni_Y \Rightarrow (b, A) \in \beta\,)$$

Kleisli's composition:

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\alpha}\boldsymbol{\beta}_{\circ}$$

Peleg's composition:

$$\boldsymbol{\alpha} * \boldsymbol{\beta} = \boldsymbol{\alpha}\boldsymbol{\beta}_{*}$$

Parikh's composition:

$$\boldsymbol{\alpha} \diamond \boldsymbol{\beta} = \boldsymbol{\alpha}\boldsymbol{\beta}_{\diamond}$$

$\boldsymbol{\beta}_{\circ}$: Kleisli lifting, $\boldsymbol{\beta}_{*}$: Peleg lifting, $\boldsymbol{\beta}_{\diamond}$: Parikh lifting

Outline

- Kleisli lifting and Kleisli's composition
- Peleg lifting and Peleg's composition
- Parikh lifting and Parikh's composition
- Associativity and the unit of each composition

Why do we have to consider the associativity?

Peleg's composition need not be associative.

Example (Furusawa and Struth, CoRR, 2014)

Let $X = \{a, b\}$, $\alpha, \beta : X \rightharpoonup \wp(X)$

$$\alpha = \{(a, \{a, b\}), (a, \{a\}), (b, \{a\})\}$$
$$\beta = \{(a, \{a\}), (a, \{b\})\}$$

Then

$$\begin{aligned} & (\alpha * \alpha) * \beta \\ = \ & \{(a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\})\} \\ \sqsubseteq \ & \{(a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\}), (a, \{a, b\})\} \\ = \ & \alpha * (\alpha * \beta) \end{aligned}$$

Why do we have to consider the associativity?

Parikh's composition need not be associative.

Example (Tsumagari, PhD thesis)

Let $X = \{a, b, c\}$, $\alpha, \beta : X \rightharpoonup \wp(X)$

$$\alpha = \{(a, \{a, b, c\}), (b, \{a, b, c\}), (c, \{a, b, c\})\}$$
$$\beta = \{(a, \{b, c\}), (b, \{a, c\}), (c, \{a, b\})\}$$

Then

$$(\alpha \diamond \beta) \diamond \alpha = 0_{X\wp(X)} \sqsubseteq \alpha = \alpha \diamond (\beta \diamond \alpha)$$

To prove the associativity

Let $\square \in \{\circ, *, \diamond\}$.

$$
\begin{array}{rrcl}
 & (\alpha \mathbin{\square} \beta) \mathbin{\square} \gamma & = & \alpha \mathbin{\square} (\beta \mathbin{\square} \gamma) \\
\leftrightarrow & (\alpha\beta_{\square}) \mathbin{\square} \gamma & = & \alpha \mathbin{\square} (\beta\gamma_{\square}) \\
\leftrightarrow & \alpha\beta_{\square}\gamma_{\square} & = & \alpha(\beta\gamma_{\square})_{\square} \\
\leftarrow & \color{red}{\beta_{\square}\gamma_{\square}} & \color{red}{=} & \color{red}{(\beta\gamma_{\square})_{\square}}
\end{array}
$$

To prove the associativity

Lemma

For $\square \in \{\circ, *, \diamond\}$,

$$\boldsymbol{\beta}_\square \boldsymbol{\gamma}_\square \sqsubseteq (\boldsymbol{\beta}\boldsymbol{\gamma}_\square)_\square$$

We have

$$(\boldsymbol{\alpha} \,\square\, \boldsymbol{\beta}) \,\square\, \boldsymbol{\gamma} \sqsubseteq \boldsymbol{\alpha} \,\square\, (\boldsymbol{\beta} \,\square\, \boldsymbol{\gamma}).$$

How about the converse implication?

Associativity of Kleisli's composition

For Kleisli's composition

Lemma

$$\boldsymbol{\beta}_\circ \boldsymbol{\gamma}_\circ = (\boldsymbol{\beta}\boldsymbol{\gamma}_\circ)_\circ$$

Proposition

$$(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ (\boldsymbol{\beta} \circ \boldsymbol{\gamma})$$

Associativity of Peleg's composition

For Peleg's composition

Lemma

If $\gamma \colon Z \rightharpoondown \wp(W)$ *is union-closed,*

$$(\beta\gamma_*)_* \sqsubseteq \beta_*\gamma_*$$

Proposition

If $\gamma \colon Z \rightharpoondown \wp(W)$ *is union-closed,*

$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$$

Associativity of Peleg's composition

Definition

$\gamma : Z \rightharpoondown \wp(W)$ is called *union-closed* if

$$\lfloor \rho \rfloor (\rho \ni_W)^{@} \sqsubseteq \gamma$$

for all relations $\rho : Z \rightharpoondown \wp(W)$ such that $\rho \sqsubseteq \gamma$.

$$(a, B) \in \alpha^{@} \Leftrightarrow B = \{b \mid (a, b) \in \alpha\}$$

Note: $\gamma : Z \rightharpoondown \wp(W)$ is union-closed iff

$$\mathcal{B} \neq \emptyset \ \wedge \ \mathcal{B} \subseteq \{B \mid \ (a, B) \in \gamma\} \Rightarrow (a, \bigcup \mathcal{B}) \in \gamma$$

for each $a \in Z$.

Associativity of Parikh's composition

For Parikh's composition

Lemma

If $\beta : Y \rightharpoondown \wp(Z)$ *is up-closed,*

$$(\beta\gamma_\diamond)_\diamond \sqsubseteq \beta_\diamond\gamma_\diamond$$

Proposition

If $\beta : Y \rightharpoondown \wp(Z)$ *is up-closed,*

$$(\alpha \diamond \beta) \diamond \gamma = \alpha \diamond (\beta \diamond \gamma)$$

Associativity of Parikh's composition

Definition

$\beta : Y \rightharpoonup \wp(Z)$ is called *up-closed* if

$$\beta \Xi_Z = \beta$$

$$(C, C') \in \Xi_Z \Leftrightarrow C \sqsubseteq C'$$

Note: $\beta : Y \rightharpoonup \wp(Z)$ is up-closed iff

$$(b, C) \in \beta \ \wedge \ C \sqsubseteq C' \rightarrow (b, C') \in \beta$$

Unit of each composition

What is the unit of each composition?

$$\alpha \,\square\, 1 = 1 \,\square\, \alpha = \alpha$$

1: the unit of $\square$

Example: multirelations on a singleton

Let $X = \{a\}$ and

$$\begin{aligned} 0 &= 0_{X\wp(X)} \\ \alpha &= \{(a, \emptyset)\} \\ \beta &= \{(a, \{a\})\} \\ \gamma &= \{(a, \emptyset), (a, \{a\})\} \end{aligned}$$

These are all relations from X to $\wp(X)$.

$\mathbf{0} = \mathbf{0}_{X\wp(X)}, \boldsymbol{\alpha} = \{(a, \emptyset)\}, \boldsymbol{\beta} = \{(a, \{a\})\}, \boldsymbol{\gamma} = \{(a, \emptyset), (a, \{a\})\}$

Kleisli liftings of these relations:

$$\mathbf{0}_\circ = \boldsymbol{\alpha}_\circ = \{(\emptyset, \emptyset), (\{a\}, \emptyset)\}$$
$$\boldsymbol{\beta}_\circ = \boldsymbol{\gamma}_\circ = \{(\emptyset, \emptyset), (\{a\}, \{a\})\}$$

Kleisli's composition table:

$\circ$	$\mathbf{0}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$
$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$
$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$

$\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are right units and there is no left unit.

If we consider mappings (i.e. total and univalent multirelations)

$$\mathbf{0} = \mathbf{0}_{X\wp(X)}, \boldsymbol{\alpha} = \{(a, \emptyset)\}, \boldsymbol{\beta} = \{(a, \{a\})\}, \gamma = \{(a, \emptyset), (a, \{a\})\}$$

Kleisli's composition table:

$\circ$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$
$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$

The singleton map $\boldsymbol{\beta}$ is the unit w.r.t. Kleisli's composition.

$$0 = 0_{X\wp(X)}, \alpha = \{(a,\emptyset)\}, \beta = \{(a,\{a\})\}, \gamma = \{(a,\emptyset),(a,\{a\})\}$$

Peleg liftings of these relations:

$$\begin{aligned}
0_* &= \{(\emptyset,\emptyset)\} \\
\alpha_* &= \{(\emptyset,\emptyset),(\{a\},\emptyset)\} \\
\beta_* &= \{(\emptyset,\emptyset),(\{a\},\{a\})\} \\
\gamma_* &= \{(\emptyset,\emptyset),(\{a\},\emptyset),(\{a\},\{a\})\}
\end{aligned}$$

Peleg's composition table:

$*$	0	α	β	γ
0	0	0	0	0
α	α	α	α	α
β	0	α	β	γ
γ	α	α	γ	γ

The singleton map β is the unit w.r.t. Peleg's composition.

[Furusawa, Struth, CoRR, 2014]

$0 = 0_{X\wp(X)}, \alpha = \{(a, \emptyset)\}, \beta = \{(a, \{a\})\}, \gamma = \{(a, \emptyset), (a, \{a\})\}$

Parikh lifting of these relations:

$$\begin{aligned} 0_\diamond &= \{(\emptyset, \emptyset), (\emptyset, \{a\})\} \\ \alpha_\diamond &= \{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \emptyset)\} \\ \beta_\diamond &= \{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \{a\})\} \\ \gamma_\diamond &= \nabla_{\wp(X)\wp(X)} \end{aligned}$$

Parikh's composition table:

$\diamond$	0	α	β	γ
0	0	0	0	0
α	γ	γ	γ	γ
β	0	α	β	γ
γ	γ	γ	γ	γ

So, β is the left unit and there is no right unit.

If we consider up-closed multirelations

$$\mathbf{0} = \mathbf{0}_{X\wp(X)}, \alpha = \{(a, \emptyset)\}, \beta = \{(a, \{a\})\}, \gamma = \{(a, \emptyset), (a, \{a\})\}$$

Parikh's composition table:

$\diamond$	$\mathbf{0}$	β	γ
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
β	$\mathbf{0}$	β	γ
γ	γ	γ	γ

$\boxed{\text{The membership relation } \beta \text{ is the unit w.r.t. Parikh's compositon.}}$

Conclusion

- We formalized 3 kinds of compositions of multirelations in relational calculi.
- We showed that each of the following subclasses of multirelations forms a category with each composition.

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mappings $f : X \to \wp(Y)$	$\alpha \circ \beta$ (Kleisli)	the singleton map $\{(a, \{a\}) \mid a \in X\}$
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Thank you for your attention!