Invariants as coreflexive bisimulations — in a coalgebraic setting

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Examples of areas of computing which have well-established, widespread theories taught in undergraduate courses:

- Parsers and compilers
- Relational databases
- Automata, labelled transition systems

This time we look into the last one in the list.
Definition 1 (by R. Milner)

(Well-known — this version taken from the Wikipedia)
A **bisimulation** is a simulation between two LTS such that its converse is also a simulation, where a **simulation** between two LTS \((X, \Lambda, \rightarrow_X)\) and \((Y, \Lambda, \rightarrow_Y)\) is a relation \(R \subseteq X \times Y\) such that, if \((p, q) \in R\), then for all \(\alpha\) in \(\Lambda\), and for all \(p' \in S\), \(p \xrightarrow{\alpha} p'\) implies that there is a \(q'\) such that \(q \xrightarrow{\alpha} q'\) and \((p', q') \in R\):

\[
\begin{array}{c}
p \xleftarrow{R} q \\
\alpha \downarrow \quad \quad \quad \alpha \\
\downarrow \quad \quad \quad \downarrow \\
p' \xleftarrow{R} q'
\end{array}
\]

Typical example of classical, descriptive definition.
Definition 2 (by Aczel & Mendler):

Given two coalgebras $c : X \rightarrow F(X)$ and $d : Y \rightarrow F(Y)$ an $F$-bisimulation is a relation $R \subseteq X \times Y$ which can be extended to a coalgebra $\rho$ such that projections $\pi_1$ and $\pi_2$ lift to $F$-comorphisms, as expressed by

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\begin{align*}
X & \xrightarrow{\pi_1} R \xrightarrow{\rho} Y \\
F \pi_1 & \downarrow \quad \quad \quad \quad F \pi_2 \\
F X & \quad \quad \quad \quad \quad \quad \quad \quad F Y
\end{align*}
```

Simpler and generic (coalgebraic)
Example: Bisimulations

Definition 3 (by Bart Jacobs):
A bisimulation for coalgebras $c : X \rightarrow F(X)$ and $d : Y \rightarrow F(Y)$ is a relation $R \subseteq X \times Y$ which is “closed under $c$ and $d$”:

$$(x, y) \in R \implies (c(x), d(y)) \in \text{Rel}(F)(R).$$

for all $x \in X$ and $y \in Y$.

($\text{Rel}(F)(R)$ stands for the relational lifting of $R$ via functor $F$.)

Still coalgebraic, pointwise — somewhat disturbed by the lifting construct — see details in [4].
Question

Are all these “the same” definition?
We will check the equivalence of these definitions by
PF-transformation
Bisimulations PF-transformed

Let us implode the outermost $\forall$ in Jacobs definition by PF-transformation:

\[
\langle \forall x, y : : x R y \Rightarrow (c x) \ Rel(F)(R) (d y) \rangle
\equiv \{ \text{PF-transform rule } (f b) R (g a) \equiv b(f^o \cdot R \cdot g)a \ \}
\langle \forall x, y : : x R y \Rightarrow x(c^o \cdot Rel(F)(R) \cdot d)y \rangle
\equiv \{ \text{drop variables (PF-transform of inclusion) } \}
R \subseteq c^o \cdot Rel(F)(R) \cdot d
\equiv \{ \text{introduce relator ; “al-djabr” rule } \}
c \cdot R \subseteq (F R) \cdot d
\equiv \{ \text{introduce Reynolds combinator } \}
c(F R \leftarrow R)d
Our PF-definition of bisimulation is similar to that presented by Roland Backhouse for dialgebras [2]: given dialgebra $F \xleftarrow{k} A \xrightarrow{R} GA$, relation $A \xleftarrow{R} A$ is a bisimulation of $k$ iff

$$GR \subseteq k \circ FR \cdot k$$

$$F A \xleftarrow{k} G A$$

$$F A \xleftarrow{k} G A$$

$$F R$$

$$G R$$

(1)
“Reynolds arrow combinator” is a relation on functions

\[ f(R \leftarrow S)g \equiv f \cdot S \subseteq R \cdot g \quad \text{cf. diagram} \]

useful in expressing properties of functions — namely \textit{monotonicity}

\[ B \leftarrow^f A \text{ is monotonic } \equiv f(\leq_B \leftarrow \leq_A)f \]

\textit{lifting}

\[ f \leq g \equiv f(\leq \leftarrow id)f \]

\textit{polymorphism (free theorem)}:

\[ G A \leftarrow^f F A \text{ is polymorphic } \equiv \langle \forall R : : f(G R \leftarrow F R)f \rangle \]

etc
Recall database projections

\[ \pi_{c,d}R \subseteq S \]
\[ \equiv \{ \text{definition given in the other talk} \} \]
\[ c \cdot R \cdot d^\circ \subseteq S \]
\[ \equiv \{ \text{functions (2nd) “al-djabr” rule} \} \]
\[ c \cdot R \subseteq S \cdot d \]
\[ \equiv \{ \text{Reynolds combinator} \} \]
\[ c(S \leftarrow R)d \]
\[ \equiv \{ \text{Reynolds combinator} \} \]
\[ c \cdot R \subseteq S \cdot d \]
\[ \equiv \{ \text{functions (1st) “al-djabr” rule} \} \]
“Al-djabr” rule for projections

\[ R \subseteq c^\circ \cdot S \cdot d \]

\[ \equiv \{ \text{introduce } \bigcirc \} \]

\[ R \subseteq \bigcirc_{c,d} S \]

Thus we get GC:

\[ \pi_{c,d} R \subseteq S \equiv R \subseteq \bigcirc_{c,d} S \quad (2) \]

In the other talk we were interested in the lower adjoint \((\pi_{c,d})\); this time we will focus on the the upper adjoint:

\[ x (\bigcirc_{c,d} S) y \equiv (c x) S (d y) \]
“Al-djabr” rule for projections

At once we get:

• $\pi_{c,d}$ and $\circ_{c,d}$ are monotonic

• Distribution properties (can be generalized to $n > 2$ arguments):

\[
\pi_{c,d}(R \cup S) = (\pi_{c,d}R) \cup (\pi_{c,d}S) \quad (3)
\]
\[
\circ_{c,d}(R \cap S) = (\circ_{c,d}R) \cap (\circ_{c,d}S) \quad (4)
\]

• etc
Why does Reynolds arrow matter?

Elegant and manageable PF-properties, eg.

\[ id \leftarrow id = id \] (5)
\[ (R \leftarrow S)^\circ = R^\circ \leftarrow S^\circ \] (6)
\[ R \leftarrow S \subseteq V \leftarrow U \iff R \subseteq V \land U \subseteq S \] (7)
\[ (R \leftarrow V) \cdot (S \leftarrow U) \subseteq (R \cdot S) \leftarrow (V \cdot U) \] (8)

as well as

\[ (f \leftarrow g^\circ)h = f \cdot h \cdot g \] (9)


These are immediately applicable to our PF version of Jacobs’ definition. For instance, (5) ensures \( id \) as bisimulation between a given coalgebra and itself (next slide):
Why Reynolds arrow matters

Calculation

\[ c(F \ id \leftarrow \ id) \equiv \{ \text{relator F preserves the identity} \} \]

\[ c(id \leftarrow id) \equiv \{ (5) \} \]

\[ c(id) \equiv \{ id \ x = x \} \]

\[ c = d \]

Too simple and obvious, even without Reynolds arrow in the play.

What about the equivalence between Jacobs’s and Aczel-Mendler’s definitions?
Why Reynolds arrow matters

To the set of known rules about *Reynolds arrow*, we add the following:

\[
\text{pair } (r, s) \text{ is a tabulation} \\
\downarrow \\
(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) = (r \leftarrow f) \cdot (s \leftarrow g)^\circ
\]  

(10)

**Tabulations**

A pair of functions \(r \quad C \quad s\) form a tabulation iff \(\langle r, s \rangle\) is injective, that is,

\[
r^\circ \cdot r \cap s^\circ \cdot s = id
\]

holds
Why Reynolds arrow matters

Example — we check that $\pi_1$ and $\pi_2$ form a tabulation:

$$\pi_1^\circ \cdot \pi_1 \cap \pi_2^\circ \cdot \pi_2 = id$$

\[\equiv \begin{cases} \text{go pointwise, where } \cap \text{ is conjunction} \end{cases}\]

\[(b, a)(\pi_1^\circ \cdot \pi_1)(y, x) \land (b, a)(\pi_2^\circ \cdot \pi_2)(y, x) \equiv (b, a) = (y, x)\]

\[\equiv \begin{cases} \text{PF-transform rule } (f \ b)R(g \ a) \equiv b(f^\circ \cdot R \cdot g)a \text{ twice} \end{cases}\]

$$\pi_1(b, a) = \pi_1(y, x) \land \pi_2(b, a) = \pi_2(y, x) \equiv (b, a) = (y, x)$$

\[\equiv \begin{cases} \text{trivia} \end{cases}\]

$$b = y \land a = x \equiv (b, a) = (y, x)$$

NB: it is a standard result that every $R$ can be factored in tabulation $R = f \cdot g^\circ$, eg. $R = \pi_1 \cdot \pi_2^\circ$. 

Jacobs ≡ Aczel & Mendler

\[ c(F R \leftarrow R) d \]

\[ \equiv \{ \text{tabulate } R = \pi_1 \cdot \pi_2^\circ \} \]

\[ c(F(\pi_1 \cdot \pi_2^\circ) \leftarrow (\pi_1 \cdot \pi_2^\circ)) d \]

\[ \equiv \{ \text{relator commutes with composition and converse } \} \]

\[ c(((F \pi_1) \cdot (F \pi_2)^\circ) \leftarrow (\pi_1 \cdot \pi_2^\circ)) d \]

\[ \equiv \{ \text{new rule (10) } \} \]

\[ c(((F \pi_1) \cdot (F \pi_2)^\circ \leftarrow (\pi_1 \cdot \pi_2^\circ)) d \]

\[ \equiv \{ \text{converse rule (6) } \} \]

\[ c((F \pi_1 \leftarrow \pi_1) \cdot (F \pi_2^\circ \leftarrow \pi_2^\circ)) d \]

\[ \equiv \{ \text{go pointwise (composition) } \} \]

\[ \langle \exists a : c(F \pi_1 \leftarrow \pi_1) a \land d(F \pi_2 \leftarrow \pi_2) a \rangle \]
Why Reynolds arrow matters

Meaning of $\langle \exists \ a : : \ c(F \pi_1 \leftarrow \pi_1)a \land d(F \pi_2 \leftarrow \pi_2)a \rangle$:

there exists a coalgebra $a$ whose carrier is the “graph” of bisimulation $R$ and which is such that projections $\pi_1$ and $\pi_2$ lift to the corresponding coalgebra morphisms.

Comments:

- One-slide-long proofs are easy to grasp
- Elegance of the calculation lies in the synergy with Reynolds arrow
- Rule (10) does most of the work — its proof is an example of generic, stepwise PF-reasoning (see this later on)
FDs on bisimulations

FD \( d \overset{R}{\rightarrow} c \) holds wherever \( R \) is a simple bisimulation from coalgebra \( d \) to coalgebra \( c \):

\[
c(F \; R \leftrightarrow \; R)\; d
\]

\[
\equiv \begin{cases} \{ \text{expand Reynolds combinator} \} \\
\end{cases}
\]

\[
c \cdot R \subseteq (F \; R) \cdot d
\]

\[
\equiv \begin{cases} \{ \text{functions (2nd) “al-djabr” rule} \} \\
\end{cases}
\]

\[
c \cdot R \cdot d^\circ \subseteq F \; R
\]

\[
\equiv \begin{cases} \{ \text{duplicate and take converses} \} \\
\end{cases}
\]

\[
c \cdot R \cdot d^\circ \subseteq F \; R \land d \cdot R^\circ \cdot c^\circ \subseteq F \; R^\circ
\]

\[
\Rightarrow \begin{cases} \{ \text{monotonicity of composition ; relators} \} \\
\end{cases}
\]

\[
c \cdot R \cdot d^\circ \cdot d \cdot R^\circ \cdot c^\circ \subseteq F(R \cdot R^\circ)
\]
FDs on bisimulations

\[ \Rightarrow \quad \{ \ R \text{ is simple ; } F \ id = id \ \} \]

\[ c \cdot R \cdot d^\circ \cdot d \cdot R^\circ \cdot c^\circ \subseteq id \]

\[ \equiv \quad \{ \ \text{FD in kernel's version} \ \} \]

\[ \ker (d \cdot R^\circ) \subseteq \ker c \]

\[ \equiv \quad \{ \ \text{FD in injectivity preorder version} \ \} \]

\[ c \leq d \cdot R^\circ \]

In other words: \( c \) can be less injective than \( d \) as far as “allowed by” \( R^\circ \) (which is injective).

So (implementation) \( d \) is allowed to distinguish states which (specification) \( c \) does not.
Invariants

Fact $c(F \text{id} \leftarrow \text{id})c$ above already tells us that $\text{id}$ is a (trivial) $F$-invariant for coalgebra $c$. In general:

F-invariants

In this setting, an $F$-invariant $\Phi$ simply is a coreflexive bisimulation between a coalgebra and itself:

$$c(F \Phi \leftarrow \Phi)c$$

(11)

Invariants bring about modalities:

$$c(F \Phi \leftarrow \Phi)c \equiv \Phi \subseteq c \circ (F \Phi) \cdot c$$

$\bigcirc_c \Phi$

cf. the “next time $X$ holds” modal operator:

$$\bigcirc_c X \overset{\text{def}}{=} c \circ (F X) \cdot c$$
Invariants — related work

Elegant PF-definition of a (relational) F-invariant already in Gibbons et al “When is a function a fold or an unfold”? [3]:

F-invariant
Given relation $F A \leftarrow A$ (a so-called F-coalgebra), we say that relation $A \leftarrow R A$ is an F-invariant for $S$ iff

$$S \cdot R \subseteq F R \cdot S$$

$$A \xrightarrow{S} FA \qquad F R \supseteq A$$

$$A \xleftarrow{R} A \xrightarrow{S} FA$$

(12)
Invariants and projections

As an upper adjoint in a Galois connection,

- \( \兴起_c \) is monotonic — thus simple proofs such as
  
  \[ \Phi \text{ is an invariant} \]
  
  \[ \equiv \ \{ \text{PF-definition of invariant} \} \]
  
  \[ \Phi \subseteq \兴起_c \Phi \]
  
  \[ \Rightarrow \ \{ \text{monotonicity} \} \]
  
  \[ \兴起_c \Phi \subseteq \兴起_c (\兴起_c \Phi) \]
  
  \[ \equiv \ \{ \text{PF-definition of invariant} \} \]
  
  \[ \兴起_c \Phi \text{ is an invariant} \]

- \( \兴起_c \) distributes over conjunction, that is PF-equality

  \[ \兴起_c (\Phi \cdot \Psi) = (\兴起_c \Phi) \cdot (\兴起_c \Psi) \]

  holds, etc
What about Milner’s original definition?

Milner’s definition is recovered via

- the power-transpose relating binary relations and set-valued functions,

\[ f = \Lambda R \equiv R = \in \cdot f \quad (13) \]

where \( A \xleftarrow{\in} \mathcal{P}A \) is the membership relation.

- the powerset relator:

\[ \mathcal{P}R = (\in \setminus (R \cdot \in)) \cap ((\in^\circ \cdot R) / (\in^\circ)) \quad (14) \]

which unfolds to an elaborate pointwise formula:

\[ Y(\mathcal{P}R)X \equiv \langle \forall a : a \in Y : \langle \exists b : b \in X : a R b \rangle \rangle \wedge \ldots \text{etc} \]
Motivation
Bisimulations
Reynolds arrow
Invariants
Summary
Proof

Calculation of Milner’s definition

c(\mathcal{P}R \leftarrow R)d
\equiv \{ \text{powerset coalgebras uniquely transpose relations} \}
(\Lambda S)(\mathcal{P}R \leftarrow R)(\Lambda U)
\equiv \{ \text{Reynolds} \}
(\Lambda S) \cdot R \subseteq (\mathcal{P}R) \cdot (\Lambda U)
\equiv \{ (14) \}
(\Lambda S) \cdot R \subseteq ((\in \setminus (R \cdot \in)) \cap ((\in^\circ \cdot R)/(\in^\circ))) \cdot (\Lambda U)
\equiv \{ \text{distribution since } \Lambda U \text{ is simple} \}
(\Lambda S) \cdot R \subseteq (\in \setminus (R \cdot \in)) \cdot (\Lambda U) \land (\Lambda S) \cdot R \subseteq ((\in^\circ \cdot R)/(\in^\circ)) \cdot (\Lambda U)
\equiv \{ \text{“al-djabr” rule (composition/division) and power transpose} \}
Calculation of Milner’s definition

\[ S \cdot R \subseteq R \cdot U \land (\Lambda S) \cdot R \subseteq ((\infty \cdot R)/(\infty)) \cdot (\Lambda U) \]

\[ \equiv \{ \text{take converses; “al-djabr” (functions)} \} \]

\[ S \cdot R \subseteq R \cdot U \land (\Lambda U) \cdot R^\circ \subseteq ((\infty \cdot R)/(\infty))^\circ \cdot (\Lambda S) \]

\[ \equiv \{ \text{divisions and power transpose} \} \]

\[ S \cdot R \subseteq R \cdot U \land U \cdot R^\circ \subseteq R^\circ \cdot S \]

Obs:

- Matteo Vaccari [6] infers the same by direct PF-transforming Milner’s original definition
- We obtain the same result by instantiating Jacobs’ definition to the power relator.
Follow up

- Further modal operators, for instance $\Box \Psi$ — henceforth $\Psi$ — usually defined as the largest invariant at most $\Psi$:

$$\Box \Psi = \langle \bigcup \Phi : : \Phi \subseteq \Psi \cap \circ_c \Phi \rangle$$

which shrinks to a greatest (post)fix-point

$$\Box \Psi = \langle \nu \Phi : : \Psi \cdot \circ_c \Phi \rangle$$

where meet (of coreflexives) is replaced by composition, as this paves the way to agile reasoning

- Properties calculated by PF-fixpoint calculation

- etc (currently writing a paper on this)
Summary

- Pointfree / pointwise dichotomy: PF is for reasoning in-the-large, PW is for the small
- Back to basics: need for computer science theory “refactoring”
- Rôle of PF-patterns: clear-cut expression of complex logic structures once expressed in less symbols
- Rôle of PF-patterns: much easier to spot synergies among different theories
- Coalgebraic approach in a relational setting: a win-win approach while putting together coalgebras (functions) + relators (relations).
- Also related: proof obligations on state invariants in VDM discharged by PF- calculation [5].
Annex — Calculation of (10)

Still need to calculate rule

\[
\text{pair } (r, s) \text{ is a tabulation}
\]

\[
\Downarrow
\]

\[
(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) = (r \leftarrow f) \cdot (s \leftarrow g)^\circ
\]

Our approach structures itself in a number of (generic) auxiliary results. First of all, and thanks to (8), only the “fission” part of the consequent of (10)

\[
(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) \subseteq (r \leftarrow f) \cdot (s \leftarrow g)^\circ
\]

calls for evidence which, for all suitably typed functions \( c \) and \( d \), equivales

\[
c \cdot f \cdot g^\circ \subseteq r \cdot s^\circ \cdot d \ \Rightarrow \ \langle \exists \ k : : \ c(r \leftarrow f)k \land d(s \leftarrow g)k \rangle
\]
\[ c \cdot f \cdot g^\circ \subseteq r \cdot s^\circ \cdot d \implies \langle \exists k : : c(r \leftarrow f)k \land d(s \leftarrow g)k \rangle \]

\[ \equiv \{ \text{“al-djabr” and Reynolds arrow} \} \]

\[ c \cdot f \subseteq r \cdot s^\circ \cdot d \cdot g \implies \langle \exists k : : c \cdot f = r \cdot k \land d \cdot g = s \cdot k \rangle \]

This, in turn, is an instance of

\[ x \subseteq r \cdot s^\circ \cdot y \implies \langle \exists k : : x = r \cdot k \land y = s \cdot k \rangle \]

\[ \equiv \{ \text{“al-djabr” and split-universal, followed by split-fusion} \} \]

\[ x \cdot y^\circ \subseteq r \cdot s^\circ \implies \langle \exists k : : \langle x, y \rangle = \langle r, s \rangle \cdot k \rangle \quad (15) \]

for \( x, y := c \cdot f, d \cdot g \), cf. diagram:

\[ \begin{align*}
A & \xleftarrow{f} B \xrightarrow{g} C \\
\downarrow{c} & \quad \quad \quad \quad \quad \quad \downarrow{d} \\
D & \xleftarrow{r} E \xrightarrow{s} F \\
\end{align*} \]
On function-split fission

The righthand side of implication (15) is an assertion of *split-fission*, an instance of function-fission in general. This can be shown to lead to two concerns:

- the image of \( \langle x, y \rangle \) must be at most the image of \( \langle r, s \rangle \) — \( \langle r, s \rangle \) “at least as surjective as” \( \langle x, y \rangle \)
- \( \langle r, s \rangle \) must be injective “relative” to \( \langle x, y \rangle \).

Concerning the former, we are happy to realize that it exactly matches the antecedent of (15):

\[
\text{img} \langle x, y \rangle \subseteq \text{img} \langle r, s \rangle \\
\equiv \quad \{ \text{split image transform, see below} \} \\
x \cdot y^\circ \subseteq r \cdot s^\circ
\]
On function-split fission

Concerning the latter, we go stronger than required in forcing $\langle r, s \rangle$ to be everywhere-injective:

$$\ker \langle r, s \rangle \subseteq id$$

$$\equiv \{ \text{kernels of splits ; kernels of functions are reflexive} \}$$

$$\ker r \cap \ker s = id$$

This is equivalent to saying that pair $r, s$ is a tabulation: thus the side condition of (10).

□
Divisibility relation on functions

\( f \div g \) iff there is a \( k \) such that

\[ g = f \cdot k \]  \hspace{1cm} (16)

holds. □

Of course, \( g \div g \) holds (\( k = id \)) and \( id \div g \) holds (\( k = g \)).

In general, to establish \( f \div g \) it is enough to find a \textit{functional} solution \( k \) to equation (16).

Clearly, a \textit{relational} upperbound for \( k \) always exists, \( f^\circ \cdot g \), cf.
Divisibility relation on functions

\( f \div g \) iff there is a \( k \) such that

\[
g = f \cdot k
\]  \hspace{1cm} (16)

holds. \( \square \)

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In general, to establish \( f \div g \) it is enough to find a functional solution \( k \) to equation (16).

Clearly, a **relational** upperbound for \( k \) always exists, \( f^\circ \cdot g \), cf.
On function fission

\[ g = f \cdot k \]
\[ \equiv \{ \text{equality of functions} \} \]
\[ f \cdot k \subseteq g \]
\[ \equiv \{ \text{“al-djabr”} \} \]
\[ k \subseteq f^\circ \cdot g \]

Let us find conditions for such a (maximal) solution \( f^\circ \cdot g \) to be a function: it must be entire

\[ id \subseteq (f^\circ \cdot g)^\circ \cdot f^\circ \cdot g \]
\[ \equiv \{ \text{“al-djabr” ; definition of image} \} \]
\[ \text{img } g \subseteq \text{img } f \]
On function fission

and simple:

\[ f^\circ \cdot g \cdot (f^\circ \cdot g)^\circ \subseteq id \]

\[ \equiv \{ \text{converses} \} \]

\[ f^\circ \cdot g \cdot g^\circ \cdot f \subseteq id \]

So, for \( f \) divides \( g \) wherever

- \( f \) at least as surjective as \( g \) and
- \( f \) “injective within the image (range) of” \( g \).

Last condition back to points: for all \( a, b \)

\[ \langle \exists \ c : : \ f \ a = g \ c = f \ b \rangle \Rightarrow a = b \]
Images of splits

Generic fact for calculating with images of splits:

\[ \text{img} \langle R, S \rangle \subseteq \text{img} \langle U, V \rangle \equiv R \cdot S^\circ \subseteq U \cdot V^\circ \quad (17) \]

Calculation:

\[ \text{img} \langle R, S \rangle \subseteq \text{img} \langle U, V \rangle \equiv \{ \text{switch to conditions} \} \]
\[ \langle R, S \rangle \cdot !^\circ \subseteq \langle U, V \rangle \cdot !^\circ \]
\[ \equiv \{ \text{“split twist” rule (18)} \} \]
\[ \langle R, ! \rangle \cdot S^\circ \subseteq \langle U, ! \rangle \cdot V^\circ \]
\[ \equiv \{ (19) \text{ thanks to } \!\text{-natural} \} \]
\[ \langle id, ! \rangle \cdot R \cdot S^\circ \subseteq \langle id, ! \rangle \cdot U \cdot V^\circ \]
\[ \equiv \{ \langle id, f \rangle \text{ is injective for any } f, \text{ thus left-cancellable} \} \]
\[ R \cdot S^\circ \subseteq U \cdot V^\circ \]
Again useful

“Split twist” rule:

$$\langle R, S \rangle \cdot T \subseteq \langle U, V \rangle \cdot X \equiv \langle R, T^\circ \rangle \cdot S^\circ \subseteq \langle U, X^\circ \rangle \cdot V^\circ$$  (18)

Conditional split-fusion:

$$\langle R, S \rangle \cdot T = \langle R \cdot T, S \cdot T \rangle \iff R \cdot (\text{img} \ T) \subseteq R \lor S \cdot (\text{img} \ T)$$  (19)
Motivation


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